Compactness Properties for some Hyperspaces and Function Spaces

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0 Preface

A main theme of this work is our interest to derive Ascoli-theorems, i.e. to establish conditions for a given family of functions, which ensure, that from a "weak"kind of compactness (resp. relative compactness or precompactness) of this family, its compactness (resp. relative compactness or precompactness) in a stronger sense, especially with respect to a natural function space structure follows. By our understanding of the essence of the various versions of Ascoli-like theorems, these conditions should not directly refer to any function-space structure, it should be possible to verify them without to know anything about a *structure*, with which the function-space may be equipped - as far as possible.

Of course, we need such properties like compactness, relative compactness and precompactness to be disposable, so we have to deal with topological structures, like topological spaces and generalized uniform structures (in the covering sense here, due to Poppe's inspiring work [39]), for instance, which we decided to investigate, both.

Concerning topological spaces, it is well known, that the structure of continuous convergence is a very suitable, very natural structure for the sets of continuous functions - even if it is not always topological, itself. But, in these cases, there is the compact-open topology for sets of functions, whose induced convergence coincides with continuous convergence, if the domain space is locally compact, and which is commonly viewed as a good "approximation" for continuous convergence. So, we focused our observations to this function-space topology (and for some cases even to more general set-open topologies).

Furthermore, we consider a kind of generalized covering spaces, called multifilter-spaces, to view as an approach to uniformity-like structures in the sense of Tukey and Poppe ([36], [38], [39]). These are built essentially similar to the kind, that Preuß ([44],[47],[48]) approaches uniform-like structures in the sense of Bourbaki (concerning entourages). The covering structures sketched here, should be understood as an attempt to extend the classical (and not unsubstantiated) distinction in descriptions of uniform structures into the realm of "convenient topology", developed by Preuß ([47],[48]), thus as a little supplement to this nice theory.

That a (partial-)covering-approach of this kind was not really done before, as far as we know, seems a little bit surprising, but may have one reason in some set-theoretical complications, resulting from the fact, that the for uniform covering-structures used (and indeed cogent) "finer"-relation looks quite unwieldy sometimes, compared with the friendly familiar inclusion of sets. Among other, chapter 1 is concerned with these problems, and especially an important connection between multifilters on a set (a key tool, defined there) and filters on its power-set is shown.

In chapter 2, we consider the categories **PFS** of powerfilter-spaces and **MFS** of multifilter-spaces and fine maps, which are essential for our (partial-)covering-approach to "convenient topology". We explain (shortly) some relations to notions like convergence, Cauchy-filter or precompactness, which are familiar from uniform spaces and which we will need here, too. It is shown, that **PFS**, **MFS** are strong topological universes and that **MFS** is concretely isomorphic to a bireflective subcategory of **PFS**. The bireflective subcategories of **MFS**, consisting of so called limited, pseudoprincipal, principal, weakly uniform or uniform multifilter-spaces, respectively, are considered. It is proved, that the subcategory of uniform principal multifilter-spaces is concretely isomorphic to the category of uniform spaces in the sense of Tukey, [49].

Chapter 3 is devoted to some useful notions for the investigations in function spaces from topological spaces and multifilter-spaces, later on. Possibly, the idea of compactoid filters could be especially mentioned from this chapter, but essentially it provides some notions and technical lemmas.

In chapter 4 we consider hyperspaces for topological spaces as well as for multifilter-spaces. Mostly emphasized are compactness properties for hit-and-miss topologies from topological spaces, simply, because they form the model, from which we will try to investigate a new approach to Ascoli-theorems in this work. Nevertheless, not all results are completely devoted to this attempt - we think, they could be interesting in their own right. There is a fairly useful set-theoretical lemma at the beginning of this chapter, for instance, and a property called "weak relative complete" is considered for subsets of topological spaces. It is a common generalization of closedness and compactness, and in fact it is exactly what is needed to get compactness from relative compactness. It is proved, that a hit-and-miss hyperspace, containing at least the nonempty closed subsets, is compact if and only if the base space is, whenever the miss-sets come from weak relative complete subsets. Furthermore, a few results on (relative) compactness of unions of (relative) compact subsets are established. Concerning hyperstructures from multifilter-spaces, we feel a quite direct transcription of the Vietoris-construction being fruitful and we give a lemma concerning precompactness of unions of precompact sets here.

The last chapter 5 is devoted to the idea, to derive Ascoli-like theorems by a very natural (almost) embedding[1] map from sets of functions between two spaces into a function-space between their hyperspaces, and applying then our knowledge on compactness (resp. relative compactness or precompactness) in these hyperspaces. Most

[1]In general, it is not an *embedding* in the strong sense, because the image needs not to be closed in the range space. But for the map, considered here, Mizokami [25] proved, that it really embeds the set of *all* continuous functions between topological spaces X, Y, if X, Y are Hausdorff. However, note that the map is almost always open, continuous and injective - a great advantage.

emphasis is given to the case of topological spaces and the compact-open topology on sets of continuous functions. The lemmas 137, 143 and theorem 147 are the key tools, allowing to produce the quite powerful Ascoli-like statements 149 and 150, which may be interesting especially, because almost none assumptions on the range space are needed, but all requirements are focused to the set of functions, whose (relative) compactness is in question, and to the sets, from which the considered set-open topology comes.

The same method, to derive Ascoli-theorems by using an (almost) embedding map into a function space between hyperspaces, is applied in the realm of limited multifilter-spaces. In this situation it is absolutely not trivial, to get the inverse of our considered map being a morphism. But at least for equiuniformly fine sets of functions and weakly uniform principal range spaces, this will hold and it leads to Ascoli-like statements 165 and 166, again. –

It is a great pleasure for me, to thank the professors Harry Poppe, Gerhard Preuß and Som Naimpally - for their very impressive and inspiring mathematical work, of course, and especially for their attentiveness, encouragement and kindness to me.
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1 Basic Concepts

1.1 Maps, Filters and Multifilters

Here we collect some set-theoretical concepts and facts, which will be needed in the chapters that follow. Some of the facts here are stated without proof - these are well known facts, and we will use them freely, without to mention this explicitly. Proofs can be found in [14], [39] or [42].

For a set X, we denote by $\mathfrak{P}(X)$ the power set of X and by $\mathfrak{P}_0(X)$ the power set without the empty set $\emptyset$.

1 Definition
A **filter** *on a set X is a nonempty subset φ of $\mathfrak{P}(X)$, which fulfills*

(1) $\emptyset \notin \varphi$

(2) $\forall A, B \in \varphi : A \cap B \in \varphi$ *and*

(3) $\forall A \in \varphi : A \subseteq B \Rightarrow B \in \varphi$.

By $\mathfrak{F}(X)$ we denote the set of all filters on the set X. If φ is a filter on a set X, then $\mathfrak{F}(\varphi)$ denotes the class of all filters ψ with $\psi \supseteq \varphi$. The maximal elements of $\mathfrak{F}(X)$ w.r.t. inclusion are called **ultrafilters**. *The set of all ultrafilters on X is denoted by $\mathfrak{F}_0(X)$, and consequently the class of all ultrafilters, which contain a filter φ is denoted by $\mathfrak{F}_0(\varphi)$.*

For a set X and a point $x \in X$ we denote by $\overset{\bullet}{x}$ the filter $\{A \subseteq X |\ x \in A\}$ on X and by $\ddot{x}$ the filter $\{\alpha \subseteq \mathfrak{P}_0(X) |\ \{x\} \in \alpha\}$ on $\mathfrak{P}_0(X)$. For abbreviation, a filter on $\mathfrak{P}_0(X)$ for a set X will be called a *powerfilter on X* . If $\mathfrak{B}$ is a subset of $\mathfrak{P}_0(X)$, s.t. $\varphi := [\mathfrak{B}] := \{A \subseteq X |\ \exists B_1, ..., B_n \in \mathfrak{B} : \bigcap_{i=1}^{n} B_i \subseteq A\}$ is a filter on X, we will call $\mathfrak{B}$ a *subbase* of this filter, and φ to be *generated* from $\mathfrak{B}$. $\mathfrak{B}$ is called a *base* of it, if even $\{A \subseteq X |\ \exists B \in \mathfrak{B} : B \subseteq A\}$ is a filter. Sometimes we will use the filter, generated from the set of all open neighbourhoods of a point x in a topological space. This is denoted by $\underline{U}(x)$.

2 Proposition
Let X, Y be sets, $f : X \to Y$ a map, $A_i, i \in I$ a family of subsets of X and $B_j, j \in J$ a family of subsets of Y. Then hold

(1) $f^{-1}(\bigcup_{j\in J} B_j) = \bigcup_{j\in J} f^{-1}(B_j)$,

(2) $f^{-1}(\bigcap_{j\in J} B_j) = \bigcap_{j\in J} f^{-1}(B_j)$,

(3) $f(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f(A_i)$,

(4) $f(\bigcap_{i\in I} A_i) \subseteq \bigcap_{i\in I} f(A_i)$,

(5) $f^{-1}(f(A_i)) \supseteq A_i$, *where equality holds, if* f *is injective,*

(6) $f(f^{-1}(B_j)) \subseteq B_j$, *where equality holds, if* f *is surjective.*

If φ is a filter on a set X and $f : X \to Y$ a map, then we mean by $f(\varphi)$ the *filter* on Y, *generated* from the images of the members of φ under f.

3 Proposition
Let X, Y *be sets,* $\varphi, \chi_i \in \mathfrak{F}(X), i \in I$ *and* $f \in Y^X$. *Then hold*

(1) $A \in f(\varphi) \Longleftrightarrow f^{-1}(A) \in \varphi$ *and*

(2) $f(\bigcap_{i\in I} \chi_i) = \bigcap_{i\in I} f(\chi_i)$.

Proof: (1): Let $A \in f(\varphi)$, then $\exists B \in \varphi : f(B) \subseteq A$. Now, $f(B) \subseteq A \Leftrightarrow B \subseteq f^{-1}(A)$, implying $f^{-1}(A) \in \varphi$, if $A \in f(\varphi)$. The other direction is clear.
(2): $A \in f(\bigcap_{i\in I} \chi_i) \Leftrightarrow f^{-1}(A) \in \bigcap_{i\in I} \chi_i \Leftrightarrow \forall i \in I : f^{-1}(A) \in \chi_i \Leftrightarrow \forall i \in I : A \in f(\chi_i) \Leftrightarrow A \in \bigcap_{i\in I} f(\chi_i)$. ■

4 Lemma
If φ *is a subbase for a filter on a set* X, *then there exists an ultrafilter* φ_0 *on* X, *which contains* φ.

5 Lemma
If φ *is a filter on a set* X, *then*

$$\varphi = \bigcap_{\psi\in\mathfrak{F}_0(\varphi)} \psi$$

holds, i.e. φ *is just the intersection of all its refining ultrafilters.*

6 Proposition
If X, Y *are sets,* $f \in Y^X$ *and* $\varphi \in \mathfrak{F}_0(X)$, *then* $f(\varphi) \in \mathfrak{F}_0(Y)$.

7 Proposition
Let X *be a set,* $\varphi \in \mathfrak{F}_0(X)$ *and* $\{A_1, ..., A_n\}$ *a finite family of subsets of* X *with* $\bigcup_{i=1}^n A_i \in \varphi$. *Then there exists a* $j \in \{1, ..., n\}$ *such that* $A_j \in \varphi$.

8 Corollary
Let X *be a set,* $\varphi_1, ..., \varphi_n \in \mathfrak{F}(X)$ *and* $\psi \in \mathfrak{F}_0(X)$ *with* $\psi \supseteq \bigcap_{i=1}^n \varphi_i$. *Then there exists an* $i \in \{1, ..., n\}$ *such that* $\psi \supseteq \varphi_i$.

9 Lemma
(Content Detector)
Let X *be a set,* $\mathfrak{A} \subseteq \mathfrak{P}(X)$ *and* $\varphi \in \mathfrak{F}(X)$. *Assume,* $\mathfrak{A}$ *is closed under finite unions of its elements. Then holds*

$$\varphi \cap \mathfrak{A} \neq \emptyset \Longleftrightarrow \forall \psi \in \mathfrak{F}_0(\varphi) : \psi \cap \mathfrak{A} \neq \emptyset \ ,$$

i.e. a filter contains an $\mathfrak{A}$*–set, iff each refining ultrafilter contains an* $\mathfrak{A}$*–set.*

Proof: Suppose $\forall\psi \in \mathfrak{F}_0(\varphi) : \exists A_\psi \in \mathfrak{A} : A_\psi \in \psi$. Now, assume $\varphi \cap \mathfrak{A} = \emptyset$. From this automatically follows $X \notin \mathfrak{A}$.
Consider $\mathfrak{B} := \{X \setminus A |\ A \in \mathfrak{A}\}$. Because of the closedness of $\mathfrak{A}$ under finite unions, $\mathfrak{B}$ is closed under finite intersection of its elements, and $\emptyset \notin \mathfrak{B}$, because $X \notin \mathfrak{A}$. For any $F \in \varphi, B \in \mathfrak{B}$ we have $F \cap B \neq \emptyset$, because $F \cap B = \emptyset$ would imply $F \subseteq X \setminus B \in \mathfrak{A}$ and therefore $\varphi \cap \mathfrak{A} \neq \emptyset$. So, $\varphi \cup \mathfrak{B}$ is a subbase of a filter and consequently, there exists an ultrafilter ψ, containing $\varphi \cup \mathfrak{B}$, therefore containing φ and the complement of every $\mathfrak{A}$–set - in contradiction to $\forall\psi \in \mathfrak{F}_0(\varphi) : \psi \cap \mathfrak{A} \neq \emptyset$. The other direction of the statement of the lemma is obvious. ■

For sets X, Y we will sometimes use the so called *evaluation map* ω, defined as

$$\omega : X \times Y^X \to Y : \omega(x, f) := f(x)$$

If $\mathcal{F}$ is a filter on Y^X and φ a filter on X, then by $\mathcal{F}(\varphi)$ we just mean $\omega(\varphi \times \mathcal{F})$, where $\varphi \times \mathcal{F}$ is the *product filter*, generated from all cartesian products of members of φ with members of $\mathcal{F}$.

10 Lemma
Let X, Y be sets, $\varphi \in \mathfrak{F}(X), \mathcal{F} \in \mathfrak{F}(Y^X)$. Then holds

$$\forall\psi \in \mathfrak{F}_0(\mathcal{F}(\varphi)) : \exists\mathcal{F}_0 \in \mathfrak{F}_0(\mathcal{F}), \varphi_0 \in \mathfrak{F}_0(\varphi) : \mathcal{F}_0(\varphi_0) \subseteq \psi\ .$$

Proof: Because of $\psi \supseteq \mathcal{F}(\varphi)$, each $C \in \psi$ has nonempty intersection with every $\omega(P \times F), P \in \varphi, F \in \mathcal{F}$, so for each $C \in \psi, P \in \varphi, F \in \mathcal{F}$, $\omega^{-1}(C) = \{(x, f) \in X \times Y^X |\ f(x) \in C\}$ has nonempty intersection with $P \times F$. Furthermore, for $C_1, C_2 \in \psi, P_1, P_2 \in \varphi, F_1, F_2 \in \mathcal{F}$ we have $\omega^{-1}(C_1) \cap \omega^{-1}(C_2) \supseteq \omega^{-1}(C_1 \cap C_2)$, which is not empty, because $C_1 \cap C_2 \in \psi$, and $(P_1 \times F_1) \cap (P_2 \times F_2) = (P_1 \cap P_2) \times (F_1 \cap F_2)$, with $P_1 \cap P_2 \in \varphi$ and $F_1 \cap F_2 \in \mathcal{F}$. Thus $\omega^{-1}(C_1) \cap (P_1 \times F_1) \cap \omega^{-1}(C_2) \cap (P_2 \times F_2) \neq \emptyset$, too. Now, $\mathfrak{B} := \{\omega^{-1}(C) \cap (P \times F) |\ C \in \psi, P \in \varphi, F \in \mathcal{F}\}$ is a filterbase on $X \times Y^X$ such that $[\mathrm{pr}_X(\mathfrak{B})] \supseteq \varphi$ and $[\mathrm{pr}_{Y^X}(\mathfrak{B})] \supseteq \mathcal{F}$, with the projection maps $\mathrm{pr}_X : X \times Y^X \to X : \mathrm{pr}_X((x, f)) := x$ and $\mathrm{pr}_{Y^X} : X \times Y^X \to Y^X : \mathrm{pr}_{Y^X}((x, f)) := f$, and $[\mathfrak{B}] \supseteq [\omega^{-1}(\psi)]$. By proposition 4, there exists an ultrafilter $\mathfrak{B}_0$ on $X \times Y^X$, which contains $\mathfrak{B}$. This implies $[\omega(\mathfrak{B}_0)] \supseteq \omega(\mathfrak{B}) \supseteq \psi$, just meaning $[\omega(\mathfrak{B}_0)] = \psi$, because ψ is an ultrafilter. Now, define $\varphi_0 := \mathrm{pr}_X(\mathfrak{B}_0), \mathcal{F}_0 := \mathrm{pr}_{Y^X}(\mathfrak{B}_0)$. By proposition 6 they are ultrafilters and we have $\varphi_0 \times \mathcal{F}_0 \subseteq \mathfrak{B}_0$, because $\forall P \times F \in \varphi_0 \times \mathcal{F}_0 : \exists B_P, B_F \in \mathfrak{B}_0 : P = \mathrm{pr}_X(B_P), F = \mathrm{pr}_{Y^X}(B_F) \Rightarrow \mathfrak{B}_0 \ni B_P \cap B_F \subseteq \mathrm{pr}_X(B_P \cap B_F) \times \mathrm{pr}_{Y^X}(B_P \cap B_F) \subseteq P \times F$. So, $\mathcal{F}_0(\varphi_0) = [\omega(\varphi_0 \times \mathcal{F}_0)] \subseteq \omega(\mathfrak{B}_0) = \psi$ follows. ■

Note, that the statement of the lemma remains true, if $\varphi, \mathcal{F}, \psi$ are powerfilters on X, Y^X, Y, respectively, because a subset of Y^X, i.e. an element F of an element of $\mathcal{F}$ in this case, works just as *one* special function from $\mathfrak{P}_0(X)$ to $\mathfrak{P}_0(Y)$ by our evaluation $F(A) := \omega(A \times F), F \subseteq Y^X, A \subseteq X$. Thus, $\mathcal{F}$ is in fact a filter on $\mathfrak{P}_0(Y)^{\mathfrak{P}_0(X)}$.

11 Corollary
Let X, Y be sets, $\varphi \in \mathfrak{F}(X)$, $f \in Y^X$ and $\psi \in \mathfrak{F}_0(Y)$ with $\psi \supseteq f(\varphi)$. Then there exists an ultrafilter $\varphi_0 \in \mathfrak{F}_0(\varphi)$ with $f(\varphi_0) = \psi$.

Proof: Choose the ultrafilter $\overset{\bullet}{f}$ as $\mathcal{F}$ in lemma 10. ■

12 Corollary
Let X, Y be sets, $\varphi \in \mathfrak{F}(X)$ and $f \in Y^X$.
Then $\mathfrak{F}_0(f(\varphi)) = f(\mathfrak{F}_0(\varphi))$ $(:= \{f(\psi) |\ \psi \in \mathfrak{F}_0(\varphi)\})$ holds.

Proof: Proposition 6 ensures $f(\mathfrak{F}_0(\varphi)) \subseteq \mathfrak{F}_0(f(\varphi))$ and from corollary 11 we get $\mathfrak{F}_0(f(\varphi)) \subseteq f(\mathfrak{F}_0(\varphi))$. ■

13 Definition
Let X be a set. We define a relation $\preceq$ on $\mathfrak{P}_0(\mathfrak{P}_0(X))$ by

$$\forall \alpha_1, \alpha_2 \in \mathfrak{P}_0(\mathfrak{P}_0(X)) : \alpha_1 \preceq \alpha_2 :\Leftrightarrow \forall A_1 \in \alpha_1 : \exists A_2 \in \alpha_2 : A_1 \subseteq A_2$$

and call α_1 **finer** *than α_2 (resp. α_2* **coarser** *than α_1), iff $\alpha_1 \preceq \alpha_2$ holds.*
If Σ_1, Σ_2 are subsets of $\mathfrak{P}_0(\mathfrak{P}_0(X))$, we call Σ_1 finer than Σ_2, iff $\forall \alpha_2 \in \Sigma_2 : \exists \alpha_1 \in \Sigma_1 : \alpha_1 \preceq \alpha_2$.

This relation is reflexive and transitive, but neither symmetric, antisymmetric nor asymmetric.

14 Definition
Let X be a set and $\alpha_1, \alpha_2 \in \mathfrak{P}_0(\mathfrak{P}_0(X))$. Then we call

$$\alpha_1 \wedge \alpha_2 := \{A_1 \cap A_2 |\ A_1 \in \alpha_1, A_2 \in \alpha_2, A_1 \cap A_2 \neq \emptyset\}$$

a **coarsest common refinement** *of α_1 and α_2.*

This operation is commutative and assoziative, so it extends naturally by recursion to finitely many operands, without respect to their ordering. The coarsest common refinement of $n \in I\!N$ partial coverings $\alpha_1, ..., \alpha_n$ of a set X we denote by $\bigwedge_{i=1}^{n} \alpha_i$. Obviously, the coarsest common refinement is indeed finer than each of the involved operands α_i. Because $\preceq$ is not antisymmetric, there are in general some more partial coverings, which are finer than all α_i and coarser than $\bigwedge_{i=1}^{n} \alpha_i$, but they are finer than $\bigwedge_{i=1}^{n} \alpha_i$ at the same time.

15 Proposition
For sets X, Y and $\alpha, \alpha_i, \beta \in \mathfrak{P}_0(\mathfrak{P}_0(X)), \gamma, \delta \in \mathfrak{P}_0(\mathfrak{P}_0(Y))$ and any function $f : X \to Y$ holds

(1) $\alpha \preceq \beta \Rightarrow f(\alpha) \preceq f(\beta)$

(2) $\gamma \preceq \delta \Rightarrow f^{-1}(\gamma) \preceq f^{-1}(\delta)$

(3) $\bigwedge_{i=1}^{n} \alpha_i \preceq f^{-1}(\bigwedge_{i=1}^{n} f(\alpha_i))$

(4) $\alpha \preceq \beta$ *and* $\gamma \preceq \delta$ *always imply* $\alpha \wedge \gamma \preceq \beta \wedge \delta$

Proof: (1): For $A' \in f(\alpha)$ we have an $A \in \alpha$ with $A' = f(A)$ and because of $\alpha \preceq \beta$, there is $B \in \beta$ such that $A \subseteq B$, which $A' = f(A) \subseteq f(B) \in f(\beta)$ implies.
(2): For $C \in f^{-1}(\gamma)$ we have $C' \in \gamma$ with $C = f^{-1}(C')$ and $D' \in \delta$ with $C' \subseteq D'$, implying $C = f^{-1}(C') \subseteq f^{-1}(D') \in f^{-1}(\delta)$.
(3): $A \in \bigwedge_{i=1}^{n} \alpha_i \Rightarrow \exists A_i \in \alpha_i, i = 1, ..., n : A = \bigcap_{i=1}^{n} A_i \Rightarrow f(A) \subseteq \bigcap_{i=1}^{n} f(A_i) \in \bigwedge_{i=1}^{n} f(\alpha_i) \Rightarrow f^{-1}(f(A)) \subseteq f^{-1}(\bigcap_{i=1}^{n} f(A_i)) \in f^{-1}(\bigwedge_{i=1}^{n} f(\alpha_i))$, and of course $A \subseteq f^{-1}(f(A))$. (4): follows from $A \subseteq B, C \subseteq D \Rightarrow A \cap C \subseteq B \cap D$. ■

In [39], Poppe deals with structures of *coverings* of a given set X, i.e. partial coverings α, which are not really partial, but fulfills $\bigcup_{A \in \alpha} A = X$. These structures are required to be directed by $\preceq$, i.e. to contain a common refinement for every pair of its members. In order to get suitable structures for our attempt to define generalized uniformities with desirable categorical properties (as natural function-spaces, for example), we will have to omit the full-covering-requirement. This seems to lead us, starting from generalized Tukey-structures, at once to the following, which we will study a little from a set-theoretical point of view, before we may try to make topological structures from this.

16 Definition
Let X be a set. A family $\Sigma \in \mathfrak{P}_0(\mathfrak{P}_0(\mathfrak{P}_0(X)))$ is called a **multifilter** *on X, iff*

(1) $\sigma_1 \in \Sigma \wedge \sigma_1 \preceq \sigma_2 \Rightarrow \sigma_2 \in \Sigma$ *and*

(2) $\sigma_1, \sigma_2 \in \Sigma \Rightarrow \exists \sigma_3 \in \Sigma : \sigma_3 \preceq \sigma_1$ *and* $\sigma_3 \preceq \sigma_2$

holds. The set of all multifilters on a set X we denote by $\widehat{\mathfrak{F}}(X)$.

In the context of condition 16(1), the condition 16(2) may be replaced equivalently by the requirement, that $\sigma_1 \wedge \sigma_2$ belongs to Σ, if σ_1 and σ_2 do. (Obviously, $\sigma_1 \wedge \sigma_2$ is finer than both, σ_1 and σ_2, so it can be chosen as the σ_3 to fulfill 16(2). Conversely, if by any $\sigma_3 \in \Sigma$ condition 16(2) is fulfilled, than $\forall S_3 \in \sigma_3 : \exists S_1 \in \sigma_1, S_2 \in \sigma_2 : S_3 \subseteq S_1 \wedge S_3 \subseteq S_2$, therefore $\forall S_3 \in \sigma_3 : \exists S_1 \in \sigma_1, S_2 \in \sigma_2 : S_3 \subseteq S_1 \cap S_2$ follows and so $\sigma_3 \preceq \sigma_1 \wedge \sigma_2$ holds. Now, $\sigma_1 \wedge \sigma_2$ belongs to Σ because of condition 16(1).)
A family Σ_1 of partial coverings is called *finer* than a family Σ_2, iff $\forall \beta \in \Sigma_2 : \exists \alpha \in \Sigma_1 : \alpha \preceq \beta$. We will write $\Sigma_1 \preceq \Sigma_2$ for this, but unless we will prefer this symbol, we should have in mind, that the statement $\Sigma_1 \preceq \Sigma_2$ is equivalent to $\Sigma_1 \supseteq \Sigma_2$, whenever Σ_1 is a multifilter, because of condition 16(1).

For a set X and a $\Xi \subseteq \mathfrak{P}_0(\mathfrak{P}_0(X))$ we denote by $[\Xi]$ the family

$$[\Xi] := \{\sigma \subseteq \mathfrak{P}_0(X) |\ \exists n \in I\!N, \xi_1, ..., \xi_n \in \Xi : \bigwedge_{i=1}^{n} \xi_i \preceq \sigma\} \ ,$$

which is either a multifilter or contains the empty set. In case, that $[\Xi]$ doesn't contain the empty set, we call Ξ a *subbase* for the *generated multifilter* $[\Xi]$.
If the family $\{\beta \in \mathfrak{P}_0(\mathfrak{P}_0(X)) | \exists \alpha \in \Xi : \alpha \preceq \beta\}$ is a multifilter, than we call Ξ a *base* of it and denote the *generated multifilter* again by $[\Xi]$.

If φ is a filter on a set X, then we denote by $\widehat{\varphi}$ the multifilter $\widehat{\varphi} := [\{\ \{A\} |\ A \in \varphi\}]$.

Let X be a set, $x \in X$ and $\alpha \subseteq \mathfrak{P}_0(X)$. Then the *star of α at x* is defined as

$$st(x, \alpha) := \bigcup_{A \in \alpha, x \in A} A \ ,$$

and the *weak star set* of α at x is defined as

$$\diamond(x, \alpha) := \{\bigcup_{i=1}^{n} A_i |\ n \in I\!N, \forall i = 1, ..., n : x \in A_i \in \alpha\} \ .$$

Furthermore, for a partial cover σ of a set X let $\sigma^{\diamond} := \bigcup_{x \in X, \diamond(x,\sigma) \neq \emptyset} \diamond(x, \sigma)$, $\sigma^* := \{st(x, \sigma) |\ x \in X, st(x, \sigma) \neq \emptyset\}$, and for a multifilter Σ on X let $\Sigma^{\diamond} := \{\xi \in \mathfrak{P}_0(\mathfrak{P}_0(X)) |\ \exists \sigma \in \Sigma : \sigma^{\diamond} \preceq \xi\}$, $\Sigma^* := \{\xi \in \mathfrak{P}_0(\mathfrak{P}_0(X)) |\ \exists \sigma \in \Sigma : \sigma^* \preceq \xi\}$.

A partial cover β of a set X is called a *barycentric refinement* of a partial cover α, iff $\beta^* \preceq \alpha$.

17 Proposition
Let $(\Sigma_i)_{i \in I}$ be a family of multifilters on a set X. Then holds

(1) $\bigcap_{i \in I} \Sigma_i = \{\bigcup_{i \in I} \alpha_i |\ \alpha_i \in \Sigma_i\}$

(2) *Let $\Sigma_1, \Sigma_2, \Xi_1, \Xi_2$ be multifilters on the same set. Then $\Sigma_1 \preceq \Xi_1, \Sigma_2 \preceq \Xi_2$ always implies $\Sigma_1 \cap \Sigma_2 \preceq \Sigma_2 \cap \Xi_2$.*

(3) *If Y is a set and $f \in Y^X$, then $[f(\bigcap_{i \in I} \Sigma_i)]_{\widehat{\mathfrak{F}}(Y)} = \bigcap_{i \in I} [f(\Sigma_i)]_{\widehat{\mathfrak{F}}(Y)}$ holds.*

(4) *Let Σ be a multifilter on X, Y a set and $f \in Y^X$. Then $f(\Sigma^{\diamond}) \preceq f(\Sigma)^{\diamond}$ and $f(\Sigma^*) \preceq f(\Sigma)^*$ hold.*

(5) *For every multifilter Σ hold $\Sigma \preceq \Sigma^{\diamond}$ and $\Sigma \preceq \Sigma^*$.*

(6) *If Σ_1, Σ_2 are multifilters with $\Sigma_1 \preceq \Sigma_2$, then $\Sigma_1^{\diamond} \preceq \Sigma_2^{\diamond}$ and $\Sigma_1^* \preceq \Sigma_2^*$ hold.*

Proof: (1): $\alpha \in \bigcap_{i\in I}\Sigma_i \Rightarrow \forall i \in I : \alpha \in \Sigma_i \Rightarrow \alpha \in \{\bigcup_{i\in I}\alpha_i |\ \alpha_i \in \Sigma_i\}$ (chose all $\alpha_i := \alpha$). Otherwise $\alpha \in \{\bigcup_{i\in I}\alpha_i |\ \alpha_i \in \Sigma_i\} \Rightarrow \forall i \in I : \exists \alpha_i \in \Sigma_i : \alpha_i \preceq \alpha \Rightarrow \forall i \in I : \alpha \in \Sigma_i \Rightarrow \alpha \in \bigcap_{i\in I}\Sigma_i$.
(2): Follows from (1), because obviously $\sigma_1 \preceq \xi_1, \sigma_2 \preceq \xi_2$ implies $\sigma_1 \cup \sigma_2 \preceq \xi_1 \cup \xi_2$.
(3): From (1) we know $[f(\bigcap_{i\in I}\Sigma_i)] = [f(\{\bigcup_{i\in I}\alpha_i |\ \alpha_i \in \Sigma_i\})] = [\{f(\bigcup_{i\in I}\alpha_i) |\ \alpha_i \in \Sigma_i\}] = [\{\bigcup_{i\in I} f(\alpha_i) |\ \alpha_i \in \Sigma_i\}] = \bigcap_{i\in I}[f(\Sigma_i)]$.
(4): Let $\sigma \in \Sigma$ be given, then always $x \in S \in \sigma$ implies $f(x) \in f(S) \in f(\sigma)$ (resp $\diamondsuit(x,\sigma) \preceq \diamondsuit(f(x), f(\sigma))$, thus $f(st(x,\sigma)) \subseteq st(f(x), f(\sigma))$ and consequently $f(\sigma^*) \preceq f(\sigma)^*$ (resp. $f(\sigma^\diamondsuit) \preceq f(\sigma)^\diamondsuit$).
(5): Follows simply from the fact, that for $S \in \sigma \in \Sigma$ with $s \in S$ always $S \subseteq st(s,\Sigma) \in \sigma^* \in \Sigma^*$ (resp. $S \subseteq S \in \diamondsuit(s,\sigma)$) holds.
(6): From $\Sigma_1 \ni \sigma_1 \preceq \sigma_2 \in \Sigma_2$ follows easily $\forall x \in X : st(x,\sigma_1) \subseteq st(x,\sigma_2)$ (resp.$\diamondsuit(x,\sigma_1) \preceq \diamond(x,\sigma_2)$), thus $\sigma_1^* \preceq \sigma_2^*$ (resp. $\sigma_1^\diamondsuit \preceq \sigma_2^\diamondsuit$). ■

18 Definition
Let $X_i, i \in I$ be sets, and $\Sigma_i \in \widehat{\mathfrak{F}}(X_i)$ for each $i \in I$. Then we call

$$\prod_{i\in I}\Sigma_i := \left[\left\{\prod_{i\in I}\sigma_i \mid \exists i_0 \in I : \sigma_{i_0} \in \Sigma_{i_0} \wedge \forall i \in I \setminus \{i_0\} : \sigma_i = \{X_i\}\right\}\right]$$

the **product of** *the* **multifilters** *$\Sigma_i, i \in I$, with $\prod_{i\in I}\sigma_i := \{\prod_{i\in I} S_i |\ \forall i \in I : S_i \in \sigma_i\}$ and $\prod_{i\in I} S_i$ means the cartesian product of sets.*

It's easy to see, that the product of multifilters is a multifilter on the cartesian product of the underlying sets - we have only to show, that the generating family of partial covers doesn't contain any finite subfamily whose coarsest common refinement is the empty set: given $\prod_{i\in I}\sigma_i^{(1)}, ..., \prod_{i\in I}\sigma_i^{(n)}$, we know, that $\prod_{i\in I}\bigwedge_{k=1}^n \sigma_i^{(k)}$ is not empty, because for all $i \in I$ and $k = 1, ..., n$ we have $\sigma_i^{(k)} \in \Sigma_i$, which is a multifilter and so $\bigwedge_{k=1}^n \sigma_i^{(k)}$ is not empty. Now, $\prod_{i\in I}\bigwedge_{k=1}^n \sigma_i^{(k)} = \{\prod_{i\in I}\bigcap_{k=1}^n S_i^{(k)} |\ S_i^{(k)} \in \sigma_i^{(k)}\}$. But for every member of this family $\prod_{i\in I}\bigcap_{k=1}^n S_i^{(k)} \subseteq \bigcap_{k=1}^n \prod_{i\in I} S_i^{(k)}$ holds, and $\bigcap_{k=1}^n \prod_{i\in I} S_i^{(k)}$ is a member of the coarsest common refinement of $\prod_{i\in I}\sigma_i^{(1)}, ..., \prod_{i\in I}\sigma_i^{(n)}$. So, this refinement is coarser than a nonempty partial cover and consequently, it's nonempty, too.

19 Definition
Let $X_i, i \in I$ be sets and $\Phi_i \in \mathfrak{F}(\mathfrak{P}_0(X_i)), i \in I$. Then we define the **product of the powerfilters** *$\Phi_i, i \in I$ by*

$$\prod_{i\in I}\Phi_i := \left[\left\{ \{A \in \mathfrak{P}_0(\prod_{i\in I} X_i) |\ p_k(A) \in \varphi_k\} \;\middle|\; k \in I, \varphi_k \in \Phi_k \right\}\right] ,$$

where $p_k : \prod_{i\in I} X_i \to X_k : (x_i)_{i\in I} \to x_k$ are the canonical projections.

20 Proposition
Let $X_i, Y_i, i \in I$ be sets, $\Phi_i \in \mathfrak{F}(\mathfrak{P}_0(X_i))$ and $f_i : X_i \to Y_i$ mappings. Then

$$(\prod_{i\in I} f_i)(\prod_{i\in I} \Phi_i) \supseteq \prod_{i\in I} f_i(\Phi_i)$$

holds. If all $f_i, i \in I$ are surjective, then

$$(\prod_{i\in I} f_i)(\prod_{i\in I} \Phi_i) = \prod_{i\in I} f_i(\Phi_i) \ .$$

Proof: We use the description of the product of the filters by suitable subbases and find $(\prod f_i)(\prod \Phi_i) = [\{\{(\prod f_i)(A)|\ A \in \mathfrak{P}_0(\prod X_i), p_k(A) \in \varphi_k\}|\ k \in I, \varphi_k \in \Phi_k\}]$ and $\prod f_i(\Phi_i) = [\{\{B \in \mathfrak{P}_0(\prod Y_i)|\ q_k(B) \in f_k(\varphi_k)\}|\ k \in I, \varphi_k \in \Phi_k\}$ with the canonical projections $p_k : \prod X_i \to X_k$ and $q_k : \prod Y_i \to Y_k$. Now, we have naturally $f_k \circ p_k = q_k \circ (\prod f_i)$, thus $p_k(A) \in \varphi_k$ implies $q_k((\prod f_i)(A)) = f_k(p_k(A)) \in f_k(\varphi_k)$, leading to $\{(\prod f_i)(A)|\ A \in \mathfrak{P}_0(\prod X_i), p_k(A) \in \varphi_k\} \subseteq \{B \in \mathfrak{P}_0(\prod Y_i)|\ q_k(B) \in f_k(\varphi_k)\}$, and consequently for the generated filters the converse relation holds.
If otherwise all f_i are surjective, then $\prod f_i$ is surjective, too. Thus, for any $B \in \mathfrak{P}_0(\prod Y_i)$ with $q_k(B) = f_k(A_k), A_k \in \varphi_k$ we have $(\prod f_i)((\prod f_i)^{-1}(B)) = B$.
We have $(\prod f_i)^{-1}(B) = \bigcup_{\underline{y}\in B}(\prod f_i)^{-1}(\underline{y})$
$= \bigcup_{\underline{y}\in B} \prod_{i\in I} f_i^{-1}(q_i(\underline{y}))$, so $p_k((\prod f_i)^{-1}(B)) = p_k(\bigcup_{\underline{y}\in B} \prod_{i\in I} f_i^{-1}(q_i(\underline{y})))$
$= \bigcup_{\underline{y}\in B} p_k(\prod_{i\in I} f_i^{-1}(q_i(\underline{y}))) = \bigcup_{\underline{y}\in B} f_k^{-1}(q_k(\underline{y})) = f_k^{-1}(q_k(B)) = f_k^{-1}(f_k(A_k)) \supseteq A_k$
and furthermore $\forall \underline{y} \in B : f_k^{-1}(q_k(\underline{y})) \cap A_k \neq \emptyset$, so by surjectivity of all f_i we can chose $x_i \in f_i^{-1}(q_i(\underline{y}))$ with especially $x_k \in A_k$, yielding $(x_i)_{i\in I} \in p_k^{-1}(A_k)$ and $(\prod f_i)((x_i)_{i\in I}) = \underline{y}$, which proves $B \subseteq (\prod f_i)(p_k^{-1}(A_k))$. Now, setting $A := (\prod f_i)^{-1}(B) \cap p_k^{-1}(A_k)$ we get $(\prod f_i)(A) = (\prod f_i)((\prod f_i)^{-1}(B) \cap p_k^{-1}(A_k))$
$= (\prod f_i)(\prod f_i)^{-1}(B) \cap (\prod f_i)(p_k^{-1}(A_k)) = B$ and $p_k(A) = p_k((\prod f_i)^{-1}(B) \cap p_k^{-1}(A_k))$
$= p_k((\prod f_i)^{-1}(B)) \cap p_k(p_k^{-1}(A_k)) = A_k$.
Thus $\{B \in \mathfrak{P}_0(\prod Y_i)|\ q_k(B) \in f_k(\varphi_k)\} \subseteq \{(\prod f_i)(A)|\ A \in \mathfrak{P}_0(\prod X_i), p_k(A) \in \varphi_k\}$ for every $k \in I, \varphi_k \in \Phi_k$, and consequently in this case the generated filters are equal. ■

21 Proposition
For arbitrary families Σ_1, Σ_2 of partial coverings on a set X and any map $f : X \to Y$ holds:

(1) $\Sigma_1 \preceq \Sigma_2 \Rightarrow f(\Sigma_1) \preceq f(\Sigma_2)$

(2) If Σ_1 is a multifilter and f surjective, then $\beta \in [f(\Sigma_1)] \Rightarrow f^{-1}(\beta) \in \Sigma_1$.

For a family of sets $X_i, Y_i, i \in I$, given multifilters $\Sigma_i \in \widehat{\mathfrak{F}}(X_i), i \in I$ and functions $f_i : X_i \to Y_i$ we have

(3) $[(\prod_{i\in I} f_i)(\prod_{i\in I}\Sigma_i)] \preceq \prod_{i\in I}[f_i(\Sigma_i)]$.
If all $f_i, i \in I$ *are surjective, then*
$[(\prod_{i\in I} f_i)(\prod_{i\in I}\Sigma_i)] = \prod_{i\in I}[f_i(\Sigma_i)]$ *holds.*

Here by $\prod_{i\in I} f_i$ *we mean the mapping from* $\prod_{i\in I} X_i$ *to* $\prod_{i\in I} Y_i$*, which is defined by* $(\prod_{i\in I} f_i)((x_i)_{i\in I}) := (f_i(x_i))_{i\in I}$.

Proof: (1) If $\beta' \in [f(\Sigma_2)]$, then there exists $\beta \in \Sigma_2$ with $f(\beta) \preceq \beta'$. By $\Sigma_1 \preceq \Sigma_2$, there is an $\alpha \in \Sigma_1$, such that $\alpha \preceq \beta$. Now, by proposition 15(1) we get $\Sigma_1 \ni f(\alpha) \preceq f(\beta) \preceq \beta'$.
(2) There are $\alpha_1, ..., \alpha_n \in \Sigma_1, n \in I\!N$ with $\bigwedge_{i=1}^n f(\alpha_i) \preceq \beta$, thus $f^{-1}(\bigwedge_{i=1}^n f(\alpha_i)) \preceq f^{-1}(\beta)$ by proposition 15(2). Now, by proposition 15(3) we have $\Sigma_1 \ni \bigwedge_{i=1}^n \alpha_i \preceq f^{-1}(\bigwedge_{i=1}^n f(\alpha_i))$, so $\bigwedge_{i=1}^n \alpha_i \preceq f^{-1}(\beta)$ by transitivity, implying $f^{-1}(\beta) \in \Sigma_1$.
(3) One subbase for the multifilter $[(\prod_{i\in I} f_i)(\prod_{i\in I}\Sigma_i)]$ consists just of the images under $(\prod_{i\in I} f_i)$ of the subbase of the product $\prod_{i\in I}\Sigma_i$, which is given in definition 18, i.e. of all partial coverings
$\big\{(\prod_{i\in I} f_i)(\prod_{i\in I}\sigma_i) \mid \exists i_0 \in I : \sigma_{i_0} \in \Sigma_{i_0} \wedge \forall i \in I \setminus \{i_0\} : \sigma_i = \{X_i\}\big\} =$
$\big\{\prod_{i\in I} f_i(\sigma_i) \mid \exists i_0 \in I : f_{i_0}(\sigma_{i_0}) \in f_{i_0}(\Sigma_{i_0}) \wedge \forall i \in I \setminus \{i_0\} : f_i(\sigma_i) = \{f_i(X_i)\}\big\}$, which are always finer than the corresponding members of the subbase for $\prod_{i\in I} f_i(\Sigma_i)$, given by definition 18, because of $\forall i \in I : \{f_i(X_i)\} \preceq \{Y_i\}$. Supposed $\forall i \in I : f_i(X_i) = Y_i$, these subbases are equal. However, even if the subfamily of all $f_i, i \in I$, which are not surjective, is at most finite, the two subbases would generate the same multifilter, because of the condition 16(2). ■

As expected from the theory of filters and ultrafilters, there are maximal elements in the set of all multifilters on a set X, too.

22 Proposition
If Σ *is a multifilter on a set* X*, then there exists a multifilter* $\Sigma' \supseteq \Sigma$ *on* X*, which is maximal w.r.t. the inclusion relation.*

Proof: We use Zorn's Lemma, so it remains only to show, that every totally ordered subset of $\widehat{\mathfrak{F}}(X)$ has an upper bound in $\widehat{\mathfrak{F}}(X)$.
Let $\mathfrak{A} \subseteq \widehat{\mathfrak{F}}(X)$ be totally ordered. We set $\Sigma_0 := \bigcup_{\Sigma\in\mathfrak{A}} \Sigma$. Then Σ_0 is a multifilter on X: Given $\alpha \in \Sigma_0, \beta \in \mathfrak{P}(\mathfrak{P}_0(X))$ with $\alpha \preceq \beta$, then there must be $\Sigma \in \mathfrak{A}$ with $\alpha \in \Sigma$, implying $\beta \in \Sigma$, because Σ is a multifilter, and consequently $\beta \in \Sigma_0$.
For $\alpha_1, \alpha_2 \in \Sigma_0$ there must be $\Sigma_1, \Sigma_2 \in \mathfrak{A}$ such that $\alpha_i \in \Sigma_i, i = 1, 2$. Because $\mathfrak{A}$ is totally ordered, $\Sigma_1 \subseteq \Sigma_2$ or $\Sigma_2 \subseteq \Sigma_1$ holds and we can assume without loss of generality, that $\Sigma_1 \subseteq \Sigma_2$. But then $\alpha_1 \in \Sigma_2$, too, implying $\emptyset \neq \alpha_1 \wedge \alpha_2 \in \Sigma_2$, because Σ_2 is a multifilter. So, $\alpha_1 \wedge \alpha_2 \in \Sigma_0$ follows, implying $\Sigma_0 \in \widehat{\mathfrak{F}}(X)$. It's obvious, that Σ_0 is an upper bound for $\mathfrak{A}$ w.r.t. inclusion. ■

By $\widehat{\mathfrak{F}}_0(X)$ we denote the set of all maximal multifilters on a set X and for a multifilter Σ we mean by $\widehat{\mathfrak{F}}_0(\Sigma)$ the set of all maximal multifilters finer than Σ.

For sets X we define operators

$$^{\cup} : \mathfrak{P}(\mathfrak{P}(X)) \to \mathfrak{P}(X) : \alpha^{\cup} := \bigcup_{A \in \alpha} A$$

and

$$^{\{\}} : X \to \mathfrak{P}(X) : x^{\{\}} := \{x\}$$

which extend naturally to

$$^{\cup} : \mathfrak{P}(\mathfrak{P}(\mathfrak{P}(X))) \to \mathfrak{P}(\mathfrak{P}(X)) : \Sigma^{\cup} := \{\sigma^{\cup} |\ \sigma \in \Sigma\}\ ,$$

$$^{\{\}} : \mathfrak{P}(X) \to \mathfrak{P}(\mathfrak{P}(X)) : A^{\{\}} := \{\{a\} |\ a \in A\}$$

and

$$^{\{\}} : \mathfrak{P}(\mathfrak{P}(X)) \to \mathfrak{P}(\mathfrak{P}(\mathfrak{P}(X))) : \varphi^{\{\}} := \{A^{\{\}} |\ A \in \varphi\}$$

23 Proposition
Let X be a set, $A \subseteq X$, $\varphi \subseteq \mathfrak{P}(X)$, $\Sigma \subseteq \mathfrak{P}(\mathfrak{P}(X))$. Then hold

(1) $(A^{\{\}})^{\cup} = A$, *i.e.* $^{\cup}$ *is a left-side*[2] *inverse for* $^{\{\}}$.

(2) $(\varphi^{\cup})^{\{\}} \preceq \varphi$

(3) $(\Sigma^{\cup})^{\{\}} \preceq \Sigma$.
Let Σ be directed by $\preceq$, then $\Sigma \preceq (\Sigma^{\cup})^{\{\}}$, iff $\exists A \subseteq X : A^{\{\}} \in \Sigma$.

(4) If φ is a filter, then $\varphi \supseteq \Sigma^{\cup} \iff \varphi^{\{\}} \preceq \Sigma$.

(5) Σ is a subbase for a multifilter, iff $\Sigma^{\cup}$ is a subbase for a filter.

(6) If Σ is a multifilter, then $\Sigma^{\cup}$ is a filter.

(7) If φ is a subbase for a filter, then $\varphi^{\{\}}$ is a subbase for a multifilter.

(8) If Σ is a maximal multifilter, then $\Sigma^{\cup}$ is an ultrafilter.

[2]It is an old, fruitless and indecided discussion between teachers in mathematics, in which order the symbols of applied operators have to arise, leading to different answers to the question, which sides inverse $^{\cup}$ is for $^{\{\}}$. Unless we will use sometimes, as just now, the exponential writing for operators (which would suggest naturally to call $^{\cup}$ a right-side inverse for $^{\{\}}$), we think of these operators to arise on the left side of some others, which are applied later than the other ones (following the "$f(x)$"-writing and suggesting naturally, too, to call our inverse mentioned a left-sided one).

Proof: (1): $a \in A \Leftrightarrow \{a\} \in A^{\{\}} \Leftrightarrow a \in (A^{\{\}})^{\cup}$.
(2): $A \in (\varphi^{\cup})^{\{\}} \Rightarrow A = \{a\}, a \in \varphi^{\cup} \Rightarrow \exists A' \in \varphi : a \in A' \Rightarrow A = \{a\} \subseteq A' \in \varphi$.
(3): $(\Sigma^{\cup})^{\{\}} \preceq \Sigma$ follows directly from (2). If Σ is directed by $\preceq$ and contains a partial covering α, which consists only of singletons, then the coarsest common refinement (and consequently every refinement) of any $\beta \in \Sigma$ with α consists only of singletons, too, and so it is finer than $\beta^{\{\}}$, thus $\Sigma \preceq (\Sigma^{\cup})^{\{\}}$. Conversely, if $\Sigma \preceq (\Sigma^{\cup})^{\{\}}$, Σ must contain a refinement of $X^{\{\}}$, which must consist only of singletons.
(4): Let $\varphi \supseteq \Sigma^{\cup}$, i.e. $\forall \sigma \in \Sigma : \sigma^{\cup} \in \varphi \Rightarrow \forall \sigma \in \Sigma : (\sigma^{\cup})^{\{\}} \in \varphi^{\{\}}$ with $(\sigma^{\cup})^{\{\}} \preceq \sigma$ by (2). Conversely, let now $\varphi^{\{\}} \preceq \Sigma$, i.e. $\forall \sigma \in \Sigma : \exists A \in \varphi : A^{\{\}} \preceq \Sigma \Rightarrow \forall a \in A : \exists A' \in \sigma : a \in A' \Rightarrow A \subseteq \sigma^{\cup}$, implying $\sigma^{\cup} \in \varphi$, because φ is a filter.
(5): Let Σ be a subbase for a multifilter, then $\forall \alpha, \beta \in \Sigma : \alpha \wedge \beta \neq \emptyset$ and consequently $\forall \alpha^{\cup}, \beta^{\cup} \in \Sigma^{\cup} : \alpha^{\cup} \cap \beta^{\cup} \neq \emptyset$. Conversely, let $\Sigma^{\cup}$ be a subbase for a filter, then $\forall \alpha^{\cup}, \beta^{\cup} \in \Sigma^{\cup} : \alpha^{\cup} \cap \beta^{\cup} \neq \emptyset$, implying $\exists A \in \alpha, B \in \beta : A \cap B \neq \emptyset$ and consequently $\alpha \wedge \beta \neq \emptyset$.
(6): Given $\alpha^{\cup}, \beta^{\cup} \in \Sigma^{\cup}$, we have $(\alpha \wedge \beta)^{\cup} = \alpha^{\cup} \cap \beta^{\cup} \in \Sigma^{\cup}$ and for $\alpha^{\cup} \in \Sigma^{\cup}, B \supseteq \alpha^{\cup}$ we find easily $\alpha \preceq \beta := \alpha \cup B^{\{\}}$, so $\beta \in \Sigma$ follows and obviously $\beta^{\cup} = B$ holds.
(7): Assume, $\varphi^{\{\}}$ is not a subbase for a multifilter. Then we have $A_1^{\{\}}, ..., A_n^{\{\}} \in \varphi^{\{\}}$ with $\bigwedge_{i=1}^n A_i^{\{\}} = \emptyset$, implying $\bigcap_{i=1}^n A_i = \emptyset$, because any existent $a \in \bigcap_{i=1}^n A_i$ would lead to $\{a\} \in \bigwedge_{i=1}^n A_i^{\{\}}$. So, φ is not a subbase for a (proper) filter. (8): If $\Sigma^{\cup}$ is not an ultrafilter, then there exists a subset $A \subseteq X$, s.t. $A \notin \Sigma^{\cup}$ and $\{A\} \cup \Sigma^{\cup}$ is a filterbase, generating the filter φ. But then $\varphi^{\{\}}$ is a subbase for a multifilter, by (7), which strictly refines Σ - in contradiction to the maximality of Σ. ■

The maximal multifilters now can be described directly in terms of usual ultrafilters:

24 Proposition
Let Σ be a multifilter on a set X. Then the following are equivalent:

(1) *Σ is a maximal multifilter,*

(2) *$\Sigma^{\cup}$ is an ultrafilter on X and $\Sigma = (\Sigma^{\cup})^{\{\}}$,*

(3) *$\exists \varphi \in \mathfrak{F}_0(X) : \Sigma = \varphi^{\{\}}$.*

Proof: We get (1)⇒(2) from proposition 23(3),(8). (2)⇒(3) holds trivially. (3)⇒(1): Let $\varphi \in \mathfrak{F}_0(X)$ with $\Sigma = \varphi^{\{\}}$ be given. For any multifilter Σ_1 with $\Sigma_1 \supseteq [\varphi^{\{\}}]$ we get $\Sigma_1^{\cup} \supseteq [\varphi^{\{\}}]^{\cup} = \varphi$, implying $\Sigma_1^{\cup} = \varphi$ by the maximality of φ. Now, for any $\alpha \in \Sigma_1$ we have $\alpha^{\cup} \in \varphi$, implying $(\alpha^{\cup})^{\{\}} \in \varphi^{\{\}}$, but $(\alpha^{\cup})^{\{\}} \preceq \alpha$ by 23(2), so $\alpha \in [\varphi^{\{\}}]$. This yields $\Sigma_1 \subseteq [\varphi^{\{\}}]$. ■

Even an analogous to lemma 10 is valid for multifilters:

25 Lemma
Let X, Y be sets, $\Sigma \in \widehat{\mathfrak{F}}(X), \mathcal{F} \in \widehat{\mathfrak{F}}(Y^X)$ and $\Xi \in \widehat{\mathfrak{F}}_0(\mathcal{F}(\Sigma))$. Then there exist $\Sigma_0 \in \widehat{\mathfrak{F}}_0(\Sigma)$ and $\mathcal{F}_0 \in \widehat{\mathfrak{F}}_0(\mathcal{F})$ such that $\mathcal{F}_0(\Sigma_0) \subseteq \Xi$ holds.

Proof: Let $\Xi \in \widehat{\mathfrak{F}}_0(\mathcal{F}(\Sigma))$ be given. Then $\Xi^\cup \in \mathfrak{F}_0(Y)$ by propositions 24 and 23(1). Furthermore $\Xi \supseteq \mathcal{F}(\Sigma)$ implies $\Xi^\cup \supseteq (\mathcal{F}(\Sigma))^\cup = \{\gamma(\sigma)^\cup | \gamma \in \mathcal{F}, \sigma \in \Sigma\} = \{\{G(S) | G \in \gamma, S \in \sigma\} | \gamma \in \mathcal{F}, \sigma \in \Sigma\} = \{\bigcup_{G\in\gamma} \bigcup_{S\in\sigma} G(S) |\ \gamma \in \mathcal{F}, \sigma \in \Sigma\} = \{(\bigcup_{G\in\gamma})(\bigcup_{S\in\sigma} S) |\ \gamma \in \mathcal{F}, \sigma \in \Sigma\} = \mathcal{F}^\cup(\Sigma^\cup)$. Now, $\mathcal{F}^\cup$ and $\Sigma^\cup$ are filters on X and Y^X, respectively, by proposition 23(6), whereas $\Xi^\cup$ is an ultrafilter on Y, so lemma 10 is applicable and ensures, that there exist ultrafilters $\mathcal{G}_0 \supseteq \mathcal{F}^\cup$ and $\varphi_0 \supseteq \Sigma^\cup$ such that $\mathcal{G}_0(\varphi_0) \subseteq \Xi^\cup$, which implies $(\mathcal{G}_0(\varphi_0))^{\{\}} \subseteq (\Xi^\cup)^{\{\}}$. Now we calculate easily $(\mathcal{G}_0(\varphi_0))^{\{\}} = \{G(P)^{\{\}} |\ G \in \mathcal{G}_0, P \in \varphi_0\} = \{\{\{g(p)\} | g \in G, p \in P\} |\ G \in \mathcal{G}_0, P \in \varphi_0\} = \{G^{\{\}}(P^{\{\}}) |\ G \in \mathcal{G}_0, P \in \varphi_0\} = \mathcal{G}_0^{\{\}}(\varphi_0^{\{\}})$, where $\mathcal{G}_0^{\{\}}$ and $\varphi_0^{\{\}}$ are maximal multifilters by proposition 24. So we choose $\mathcal{F}_0 := \mathcal{G}_0^{\{\}}$ and $\Sigma_0 := \varphi_0^{\{\}}$ and find $\mathcal{F}_0(\Sigma_0) \subseteq (\Xi^\cup)^{\{\}}$, where $(\Xi^\cup)^{\{\}} = \Xi$ by proposition 23(2) and the maximality of Ξ.■

Nevertheless, in contrast to usual filters, there is a gap in the relation between multifilters and their refining maximal multifilters.

26 Lemma
If X is a set, then holds

$$\bigcap_{\Xi\in\widehat{\mathfrak{F}}_0(\Sigma)} \Xi = (\Sigma^\cup)^{\{\}}$$

for every multifilter Σ on X.

Proof: $(\Sigma^\cup)^{\{\}} \subseteq \bigcap_{\Xi\in\widehat{\mathfrak{F}}_0(\Sigma)} \Xi$ holds trivially because every $\Xi \in \widehat{\mathfrak{F}}_0(\Sigma)$ contains Σ, implying $\Xi^\cup \supseteq \Sigma^\cup$ and thus $\Xi = (\Xi^\cup)^{\{\}} \supseteq (\Sigma^\cup)^{\{\}} \supseteq \Sigma$.
If otherwise a partial covering by singletons, given without loss of generality as $A^{\{\}}, A \subseteq X$ belongs not to $(\Sigma^\cup)^{\{\}}$, then $A \notin \Sigma^\cup$ follows, and an ultrafilter φ must exist on X which contains both, $\Sigma^\cup$ and $A^c = X \setminus A$. But then $[\varphi^{\{\}}]$ is a maximal multifilter, containing $(\Sigma^\cup)^{\{\}}$ and $(A^c)^{\{\}}$, so not containing $A^{\{\}}$, which is consequently not a member of $\bigcap_{\Xi\in\widehat{\mathfrak{F}}_0(\Sigma)} \Xi$. Thus $\Sigma^{\{\}} \supseteq \bigcap_{\Xi\in\widehat{\mathfrak{F}}_0(\Sigma)} \Xi$ holds. ■

This means, the intersection of all refining maximal multifilters of a multifilter Σ is not necessary equal to the given multifilter Σ, but to a (in general: proper) refinement of it. So, a lot of multifilters have the same intersection of their refining maximal multifilters, thus a multifilter is *not* determined by its set of refining maximal multifilters. One may feel this like a structural defect of these objects, especially having in mind some kind of "pseudo-"(topological, uniform or generalized uniform) structures, to define by using multifilters as generalization of Morita's and Poppe's covering structures (based on Tukey's description of uniformity).

Thus, at a first view, the $\preceq$-relation seems to be not even convenient to build the basic objects for defining generalized uniform structures in a (partial-) covering sense. We will see later, that this relation is useful to define and to simplify some kind of generalized covering-uniformity-structures, and the notion of a multifilter is

not defined here to vanish now - they will come back again. But we should have in mind, that they will be used mostly as abbreviations for some sets of powerfilters, like suggested by the following.

27 Proposition
Let X be a set, $\Sigma \in \widehat{\mathfrak{F}}(X)$ and $\widehat{\varphi} \in \mathfrak{F}(\mathfrak{P}_0(X))$. Then hold

(1) $\Sigma^{\mathfrak{P}_0} := \{\bigcup_{A\in\alpha} \mathfrak{P}_0(A) |\ \alpha \in \Sigma\}$ *is a base for a filter* $[\Sigma^{\mathfrak{P}_0}]_{\mathfrak{F}(\mathfrak{P}_0(X))}$ *on* $\mathfrak{P}_0(X)$.

(2) $\widehat{\varphi}$ *is a base for a multifilter* $[\widehat{\varphi}]_{\widehat{\mathfrak{F}}(X)}$ *on* X.

(3) $[[\Sigma^{\mathfrak{P}_0}]_{\mathfrak{F}(\mathfrak{P}_0(X))}]_{\widehat{\mathfrak{F}}(X)} = \Sigma$

(4) $\widehat{\varphi} \preceq [\widehat{\varphi}]_{\widehat{\mathfrak{F}}(X)}$ *and* $[\widehat{\varphi}]_{\widehat{\mathfrak{F}}(X)} \preceq \widehat{\varphi}$, *as well as*

(5) $[\Sigma^{\mathfrak{P}_0}]_{\mathfrak{F}(\mathfrak{P}_0(X))} \preceq \Sigma$ *and* $\Sigma \preceq [\Sigma^{\mathfrak{P}_0}]_{\mathfrak{F}(\mathfrak{P}_0(X))}$, *compared just as subsets of* $\mathfrak{P}(\mathfrak{P}_0(X))$, *according to definition 13.*

Proof: (1) For $\Sigma \in \widehat{\mathfrak{F}}(X), \alpha, \beta \in \Sigma$ we find $(\bigcup_{A\in\alpha} \mathfrak{P}_0(A)) \cap (\bigcup_{B\in\beta} \mathfrak{P}_0(B)) = \{M \in \mathfrak{P}_0(X) |\ \exists A \in \alpha, B \in \beta : M \subseteq A \wedge M \subseteq B\} = \{M \in \mathfrak{P}_0(X) |\ \exists C \in \alpha \wedge \beta \in \Sigma : M \subseteq C\} = \bigcup_{C\in\alpha\wedge\beta} \mathfrak{P}_0(C)$.
(2) For $\alpha, \beta \in \widehat{\varphi}$ we have obviously $\emptyset \neq \alpha \cap \beta \preceq \alpha \wedge \beta$.
(3) Let $\alpha \in \Sigma$, then $\bigcup_{A\in\alpha} \mathfrak{P}_0(A) \in [\Sigma^{\mathfrak{P}_0}]_{\mathfrak{F}(\mathfrak{P}_0(X))}$ and by $\bigcup_{A\in\alpha} \mathfrak{P}_0(A) \preceq \alpha$ we get $\alpha \in [[\Sigma^{\mathfrak{P}_0}]_{\mathfrak{F}(\mathfrak{P}_0(X))}]_{\widehat{\mathfrak{F}}(X)}$. If otherwise $\alpha \in [[\Sigma^{\mathfrak{P}_0}]_{\mathfrak{F}(\mathfrak{P}_0(X))}]_{\widehat{\mathfrak{F}}(X)}$, then there must be an $\alpha' \in [\Sigma^{\mathfrak{P}_0}]_{\mathfrak{F}(\mathfrak{P}_0(X))}$ with $\alpha' \preceq \alpha$ and consequently an $\alpha'' \in \Sigma$ such that $\bigcup_{A\in\alpha''} \mathfrak{P}_0(A) \subseteq \alpha'$ holds. By $\alpha'' \preceq \bigcup_{A\in\alpha''} \mathfrak{P}_0(A)$ now $\alpha'' \preceq \alpha' \preceq \alpha$ and consequently $\alpha \in \Sigma$ follow.
(4) $\widehat{\varphi} \preceq [\widehat{\varphi}]_{\widehat{\mathfrak{F}}(X)}$ follows immediately from $\forall \alpha, \beta \in \widehat{\varphi} : \widehat{\varphi} \ni \alpha \cap \beta \preceq \alpha \wedge \beta$ and $[\widehat{\varphi}]_{\widehat{\mathfrak{F}}(X)} \preceq \widehat{\varphi}$ holds trivially because of $[\widehat{\varphi}]_{\widehat{\mathfrak{F}}(X)} \supseteq \widehat{\varphi}$.
(5) For every $\alpha \in \Sigma$, there is $\bigcup_{A\in\alpha} \mathfrak{P}_0(A)$ an element of $[\Sigma^{\mathfrak{P}_0}]_{\mathfrak{F}(\mathfrak{P}_0(X))}$, which is obvious finer than α and coarser than α, too. Both relations between Σ and $[\Sigma^{\mathfrak{P}_0}]_{\mathfrak{F}(\mathfrak{P}_0(X))}$ follow. ■

1.2 Categorical Basics

Here we will provide only a few definitions, mostly concerning desirable properties of categories in topology. For really good and motivating explanations to this topic, read [43] and [48]. For a quick overview, see the introduction of [7].

28 Definition
A concrete category $\mathcal{C}$ is said to be **topological**, *iff*

(1) fibre-smallness: For every set X the class of all $\mathcal{C}$-objects with underlying set X is a set.

(2) *terminal separator property:* For every set X with cardinality at most one, there is precisely one $\mathcal{C}$-object with underlying set X.

(3) *initial completeness:* For any set X and any indicated class $(\mathbf{X}_i, f_i : X \to X_i)_{i\in I}$ of $\mathcal{C}$-objects $\mathbf{X}_i$ with underlying sets X_i and maps f_i from X, there exists a unique $\mathcal{C}$ object with underlying set X, which is initial w.r.t. $(X, (\mathbf{X}_i, f_i : X \to X_i)_{i\in I})$, i.e. such that for any $\mathcal{C}$-object $\mathbf{Y}$ with underlying set Y, a map $g : Y \to X$ is a $\mathcal{C}$-morphism from $\mathbf{Y}$ to $\mathbf{X}$, iff for all $i \in I$ the composite maps $f_i \circ g$ are $\mathcal{C}$-morphisms from $\mathbf{Y}$ to $\mathbf{X}_i$, respectively.

A category $\mathcal{C}$ is called **cartesian closed**, iff

(4) (a) For every pair (A, B) of $\mathcal{C}$-objects exists a product $A \times B$ in $\mathcal{C}$ and

(b) For every pair (A, B) of $\mathcal{C}$-objects exists a $\mathcal{C}$-object B^A and a $\mathcal{C}$-morphism $e : A \times B^A \to B$, s.t. for every $\mathcal{C}$-Object C and every $\mathcal{C}$-morphism $f : A \times C \to B$ there exists a unique $\mathcal{C}$-morphism $\overline{f} : C \to B^A$ with $f = e \circ (\mathbb{1}_A \times \overline{f})$.

A topological category $\mathcal{C}$ is said to be **extensional**, iff for every $\mathbf{Y} \in |\mathcal{C}|$ with underlying set Y, there exists a $\mathcal{C}$-object $\mathbf{Y}^*$ with underlying set $Y^* := Y \cup \{\infty_Y\}$, $\infty_Y \notin Y$, s.t. for every $\mathbf{X} \in \mathcal{C}$ with underlying set X, every $Z \subseteq X$ and every $f : Z \to Y$, where f is a $\mathcal{C}$-morphism w.r.t. the subobject $\mathbf{Z}$ of $\mathbf{X}$ on Z, the map $f^* : X \to Y^*$, defined by

$$f^*(x) := \begin{cases} f(x) & ; \quad x \in Z \\ \infty_Y & ; \quad x \notin Z \end{cases}$$

is a $\mathcal{C}$-morphism.
A topological category $\mathcal{C}$ is called a **topological universe**, iff it is cartesian closed and extensional. It is called a **strong topological universe**, iff in addition all products of quotient maps are quotient maps in $\mathcal{C}$.

29 Definition
Let $\mathcal{C}$ be a topological category, $\mathbf{X} \in \mathcal{C}$ with underlying set X and let $Y \subseteq X$. Then we denote the initial structure on Y w.r.t. $(\mathbf{X}, i : Y \to X)$ with the canonical injection $i : Y \to X : y \to y$ as **the canonical $\mathcal{C}$–subspace structure**[3] on Y w.r.t. $\mathbf{X}$.

30 Definition
A full and isomorphism-closed subcategory $\mathcal{A}$ of a topological category $\mathcal{C}$ is called **bireflective in $\mathcal{C}$**, iff for each $\mathcal{C}$-object $\mathbf{X}$ with underlying set X there exists an $\mathcal{A}$-object $\mathbf{X}'$ with the same underlying set, such that $\mathbb{1}_X : \mathbf{X} \to \mathbf{X}'$ is a $\mathcal{C}$-morphism and for every $\mathcal{A}$-object $\mathbf{Y}$ holds $[\mathbf{X}, \mathbf{Y}]_{\mathcal{C}} \subseteq [\mathbf{X}', \mathbf{Y}]_{\mathcal{A}}$ $(= [\mathbf{X}', \mathbf{Y}]_{\mathcal{C}}$, because $\mathcal{A}$ is a full

[3]To use the notion *subspace* is justified, as far as [43], 1.2.2.5 ensures, that this structure yields a subobject in the sense of [43] 1.2.2.6.

subcategory.)
$\mathcal{A}$ is called **bicoreflective in $\mathcal{C}$**, *iff for each $\mathcal{C}$-object $\mathbf{X}$ with underlying set X there exists an $\mathcal{A}$-object $\mathbf{X}'$ with the same underlying set, such that $\mathbb{1}_X : \mathbf{X}' \to \mathbf{X}$ is a $\mathcal{C}$-morphism and for every $\mathcal{A}$-object $\mathbf{Y}$ holds $[\mathbf{Y}, \mathbf{X}]_{\mathcal{C}} \subseteq [\mathbf{Y}, \mathbf{X}']_{\mathcal{A}}$.*

Every bireflective and every bicoreflective (full and isomorphism-closed) subcategory of a topological category is topological ([43], Th. 2.2.12). Intersections of bireflective subcategories are bireflective, too ([7], Cor. 0.2.7.).

1.3 Convergence Structures

A convergence structure on a set X is a subset $q \subseteq \mathfrak{F}(X) \times X$, s.t. $\forall x \in X : (\overset{\bullet}{x}, x) \in q$ and $\forall \varphi, \psi \in \mathfrak{F}(X), x \in X : (\varphi, x) \in q \wedge \psi \supseteq \varphi \Rightarrow (\psi, x) \in q$. The pair (X, q) is called a convergence space. For more explanations, see [48] or [16], [39]. Convergence structures can be derived in well known and usual manners from other topological structures, as topologies, uniformities or bornologies, for instance.

31 Definition
Let (X, q) be a generalized convergence space. A subset $M \subseteq X$ is called **relative compact in** *X, iff*
$\forall \varphi \in \mathfrak{F}_0(M) : \exists x \in X : (\varphi, x) \in q$ holds. It is called **compact**, *iff*
$\forall \varphi \in \mathfrak{F}_0(M) : \exists m \in M : (\varphi, m) \in q$ holds.

32 Definition
A convergence space (X, q) is called a **Kent**-*convergence space, iff $(\varphi, x) \in q$ always implies $(\varphi \cap \overset{\bullet}{x}, x) \in q$.*
A convergence space is said to be

(1) *R_0 (or a R_0–space or* **symmetric**), *iff*
$\forall x, y \in X, \varphi \in \mathfrak{F}(X) : (\varphi, x) \in q \wedge \overset{\bullet}{y} \supseteq \varphi \;\Rightarrow\; (\varphi, y) \in q.$

(2) *T_0 (or a T_0-space), iff* $\forall x, y \in X : (\overset{\bullet}{x}, y) \in q \wedge (\overset{\bullet}{y}, x) \in q \Rightarrow x = y,$

(3) *T_1 (or a T_1-space), iff* $\forall x, y \in X : (\overset{\bullet}{x}, y) \in q \vee (\overset{\bullet}{y}, x) \in q \Rightarrow x = y,$

(4) *T_2 (or a Hausdorff-space), iff*
$\forall x, y \in X, \varphi \in \mathfrak{F}(X) : (\varphi, y) \in q \wedge (\varphi, x) \in q \Rightarrow x = y.$

33 Proposition
A symmetric Kent-convergence space (X, q) is T_0 if and only if it is T_1.

Proof: Let (X, q) be T_0 and $(\overset{\bullet}{x}, y) \in q$. Then from the Kent-property follows $(\overset{\bullet}{x} \cap \overset{\bullet}{y}, y) \in q$ and then from the symmetry $(\overset{\bullet}{x} \cap \overset{\bullet}{y}, x) \in q$, because $\overset{\bullet}{x} \supseteq \overset{\bullet}{x} \cap \overset{\bullet}{y}$, and consequently $(\overset{\bullet}{y}, x) \in q$, thus from T_0 follows $x = y$. The other direction is trivial.■

34 Proposition
For a topological space (X,τ) are equivalent:

(1) $\forall x \in X, U \in \underline{U}(x) : \exists V \in \tau : x \notin V \wedge V \supseteq U^c$.

(2) $\forall x, y \in X : (\overset{\bullet}{x}, y) \in q_\tau \Longrightarrow (\overset{\bullet}{y}, x) \in q_\tau$

(3) $\forall x, y \in X : \ (\overset{\bullet}{x}, y) \in q_\tau \Longrightarrow \underline{U}(x) = \underline{U}(y)$.

(4) (X, q_τ) *is* R_0*–space*

Proof: "(1)⇒(2)": Let $x, y \in X$ and assume $(\overset{\bullet}{x}, y) \in q_\tau$. Then $\overset{\bullet}{x} \supseteq \underline{U}(y)$, thus $\forall U \in \underline{U}(y) : x \in U$. For each $V \in \underline{U}(x)$ now by (1) follows: $\exists W \in \tau : x \notin W \wedge W \supseteq V^c$. This implies $y \notin V^c \subseteq W$, because otherwise there would hold $x \notin W \in \underline{U}(y)$ - in contradiction to our assumption. Thus $y \in V$, and we have $\overset{\bullet}{y} \supseteq \underline{U}(x)$, i.e. $(\overset{\bullet}{y}, x) \in q_\tau$.
"(2)⇒(3)": Let $(\overset{\bullet}{x}, y) \in q_\tau \Rightarrow \underline{U}(y) = \overset{\bullet}{y} \cap \tau \subseteq \overset{\bullet}{x} \Rightarrow \underline{U}(y) = \overset{\bullet}{y} \cap \tau \subseteq \overset{\bullet}{x} \cap \tau = \underline{U}(x)$. By (2) we have $(\overset{\bullet}{y}, x) \in q_\tau$, too, implying $\underline{U}(x) \subseteq \underline{U}(y)$ in the same manner.
"(3)⇒(1)": Let $U \in \underline{U}(x)$. Then exists $U_0 \in \overset{\bullet}{x} \cap \tau$ s.t. $U_0 \subseteq U$. Now holds

$$\begin{array}{lllll} \forall y \in U_0^c & : & U_0 \notin \underline{U}(y) & \Longrightarrow & \underline{U}(y) \not\supseteq \underline{U}(x) \\ & : & & \Longrightarrow & \underline{U}(x) \not\supseteq \underline{U}(y) \\ & & & & \text{because otherwise } (\overset{\bullet}{x}, y) \in q_\tau \text{would hold,} \\ & & & & \text{implying } \underline{U}(x) = \underline{U}(y) \text{by (3)} \\ \Rightarrow \forall y \in U_0^c & : & \exists V_y \in \underline{U}(y) \cap \tau & : & x \notin V_y \end{array}$$

With these V_y define $W := \bigcup\{V_y \mid y \in U_0^c\}$ and see easily $W \in \tau$, because $\forall y \in U_0^c : V_y \in \tau$, $x \notin W$, because $\forall y \in U_0^c : x \notin V_y$ and $U_0^c \subseteq W$.
"(3)⇒(4)": If $(\varphi, x) \in q_\tau$ then $\varphi \supseteq \underline{U}(x)$ and if $\overset{\bullet}{y} \supseteq \varphi$, then $(\overset{\bullet}{y}, x) \in q_\tau$, too. But then $\varphi \supseteq \underline{U}(y) = \underline{U}(x)$ by (3), just meaning $(\varphi, y) \in q_\tau$.
"(4)⇒(3)": Assume $(\overset{\bullet}{x}, y) \in q_\tau$, which naturally implies $\overset{\bullet}{x} \supseteq \underline{U}(y)$ and $\underline{U}(x) \supseteq \underline{U}(y)$. Trivially we have $(\underline{U}(y), y) \in q_\tau$. Now, with $\overset{\bullet}{x} \supseteq \underline{U}(y)$ and R_0 follows $(\underline{U}(y), x) \in q_\tau$, implying $\underline{U}(y) \supseteq \underline{U}(x)$. ■

35 Definition
Let (X, q) be a convergence space and φ a filter on X. A point $x \in X$ is called an **adherence point** *of φ, iff a refining ultrafilter of φ exists, which converges to x. The set*

$$adh(\varphi) := q(\mathfrak{F}_0(\varphi)) = \{x \in X \mid \exists \varphi_0 \in \mathfrak{F}_0(\varphi) : (\varphi_0, x) \in q\}$$

is called the **adherence** *of the filter φ.*
By the **adherence of a subset** *A of X we mean the adherence of the principal filter $[A]$ and call it the* **closure** *of A.*

36 Proposition
Let (X,τ) be a topological space and φ a filter on X. Then holds

$$adh(\varphi) = \bigcap_{A\in\varphi} \overline{A}$$

and especially, $adh(\varphi)$ itself is closed w.r.t. τ.

Proof: Let $x \in adh(\varphi)$. Then $\exists\varphi_0 \in \mathfrak{F}_0(\varphi) : (\varphi_0, x) \in q_\tau$ holds, implying $\forall A \in \varphi : x \in \overline{A}$, thus $x \in \bigcap_{A\in\varphi} \overline{A}$.
Otherwise let $x \in \bigcap_{A\in\varphi} \overline{A}$. Then we have $\forall A \in \varphi, U \in \overset{\bullet}{x} \cap \tau : \overline{A} \cap U \neq \emptyset$ and consequently $A \cap U \neq \emptyset$, because of the closedness properties. Thus, the family $\mathfrak{B} := \{U \cap A |\ U \in \overset{\bullet}{x} \cap \tau, A \in \varphi\}$ is a base for a filter, which refines φ and converges to x, implying $x \in adh(\varphi)$. ■

37 Definition
If (X,τ) is a topological space, we get a relation $\tilde{q_\tau}$ on $\mathfrak{F}(X)$ determined by

$$(\varphi,\psi) \in \tilde{q_\tau} :\Leftrightarrow \varphi \supseteq \psi \cap \tau\ .$$

We call it the **filter valued quasiorder** *induced by τ.*

Obviously, for a singleton-filter $\overset{\bullet}{x}$ on X and $\varphi \in \mathfrak{F}(X)$, the statement $(\varphi, \overset{\bullet}{x}) \in \tilde{q_\tau}$ just means, that φ converges to x.

1.4 Uniform Covering Structures

In [39] Poppe defines a *generalized uniform space* on a set X just as an ordered pair (X,Σ) of the set X and a multifilter-*base* Σ on X, consisting only of full covers of X. So, by omitting the requirement to contain with a cover all coarser covers, too, he gets the possibility, to define a topology from this structure in a convenient manner and to extensively study this: τ_Σ is taken as the topology, generated from the subbase $\bigcup_{\sigma\in\Sigma} \sigma$. (It is obvious, that this would lead always to the discrete topology, if arbitrary coarser covers would be required.) Another topology is regarded, too, which is more independent of a special base, but sometimes a little rough: $\tilde{\tau}_\Sigma$ consists of all sets $O \subseteq X$ for which $\forall x \in O : \exists\sigma \in \Sigma : st(x,\sigma) \subseteq O$.

To - possibly - be a convenient kind of generalization of *uniformities* in our opinion, topological structures should fulfill the following basic requirements (besides the condition of containing a subclass equivalent to the classical uniformities): there is a possibility to define a *Cauchy-property* (for filters, nets or other objects) and *uniformly continuous functions*, which both lead back to the classical notion on the subclass equivalent to the classical uniformities. Furthermore, a relationship to convergence-structures should exist, such that

(1) convergence implies Cauchy–property and

(2) uniform continuity implies continuity of a function.

This is a motivation to derive an additional convergence-structure q_Σ from generalized uniform spaces in the sense of [39] - just between the topologies τ_Σ and $\tilde{\tau}_\Sigma$ as defined in [39] -, which seems to be more suitable, because it admits to realize both of our requirements above in a natural way, whereas the topology τ_Σ is too strong for (2), and $\tilde{\tau}_\Sigma$ is too weak for (1), in general.

38 Definition
Let Σ be a family of partial coverings on X. A filter $\varphi \in \mathfrak{F}(X)$ is called a **Σ–Cauchy–filter**, *iff $\forall \alpha \in \Sigma : \varphi \cap \alpha \neq \emptyset$ holds. The set of all Σ–Cauchy–filters on X we denote by γ_Σ.*

39 Proposition
For any family Σ of coverings on a set X hold

(a) $\varphi \in \gamma_\Sigma \wedge \psi \supseteq \varphi \Longrightarrow \psi \in \gamma_\Sigma$ *und*

(b) $\forall x \in X : \overset{\bullet}{x} \in \gamma_\Sigma$.

Proof: (1) $\forall \alpha \in \Sigma : \psi \cap \alpha \supseteq \varphi \cap \alpha \neq \emptyset$
(2) $\forall \alpha \in \Sigma : x \in \bigcup \alpha = X \Rightarrow \exists A \in \alpha : x \in A \Rightarrow A \in \overset{\bullet}{x}$. ■

40 Definition
Let Σ be a family of coverings of a set X. Then the convergence structure
$q_\Sigma := \{(\varphi, x) \in \mathfrak{F}(X) \times X \mid \forall \alpha \in \Sigma : \exists A \in \alpha : x \in A \in \varphi\}$
is called the symmetric Σ-uniform *convergence on X.*

41 Lemma
With Σ a family of coverings of a set X hold:

(1) $(\varphi, x) \in q_\Sigma \Leftrightarrow \varphi \cap \overset{\bullet}{x} \in \gamma_\Sigma$

(2) $q_{\tilde{\tau}_\Sigma} \supseteq q_\Sigma \supseteq q_{\tau_\Sigma}$

(3) $q_\Sigma^{-1}(X) \subseteq \gamma_\Sigma$, *i.e. every q_Σ–convergent filter is a Σ–Cauchy–filter.*

(4) (X, q_Σ) *is a R_0–space.*

Furthermore we have

(5) *If $(X, \tau) \in |\,\mathbf{TOP}\,|$, then there exists a covering system Σ on X with $q_\Sigma = q_\tau$, if and only if (X, τ) is a R_0–space.*

Proof: (1) By $x \in A \in \varphi \Leftrightarrow A \in \varphi \cap \overset{\bullet}{x}$ follows $(\forall \alpha \in \Sigma : \exists A \in \alpha : x \in A \in \varphi) \Longleftrightarrow \left(\forall \alpha \in \Sigma : \alpha \cap (\varphi \cap \overset{\bullet}{x}) \neq \emptyset\right)$, which is by definition 38 equivalent to $\varphi \cap \overset{\bullet}{x} \in \gamma_\Sigma$.
(2) Let $(\varphi, x) \in q_{\tau_\Sigma}$, i.e. $\forall \alpha \in \Sigma : \overset{\bullet}{x} \cap \alpha \subseteq \varphi$. Now, each $\alpha \in \Sigma$ covers X, so $\overset{\bullet}{x} \cap \alpha \neq \emptyset$ follows, implying $\forall \alpha \in \Sigma : \emptyset \neq \overset{\bullet}{x} \cap \alpha = \overset{\bullet}{x} \cap (\overset{\bullet}{x} \cap \alpha) \cap \alpha \subseteq (\overset{\bullet}{x} \cap \varphi) \cap \alpha$, yielding $(\varphi, x) \in q_\Sigma$ by (1). Let now $(\varphi, x) \in q_\Sigma$ and any $\tilde{\tau}_\Sigma$-open neighbourhood O of x be given, i.e. $\exists \sigma \in \Sigma : st(x, \sigma) \subseteq O$, yielding $\forall S \in \sigma \cap \overset{\bullet}{x} : S \subseteq O$, but there exists at least one $S \in \sigma \cap \overset{\bullet}{x}$, which is an element of φ, too, because of $(\varphi, x) \in q_\Sigma$, implying $O \in \varphi$. This is valid for all $\tilde{\tau}_\Sigma$-open neighbourhoods of x, thus $(\varphi, x) \in q_{\tilde{\tau}_\Sigma}$.
(3) $\varphi \in q_\Sigma^{-1}(X) \Rightarrow \exists x \in X : (\varphi, x) \in q_\Sigma \overset{(1)}{\Rightarrow} \varphi \cap \overset{\bullet}{x} \in \gamma_\Sigma$. From proposition 39 and $\varphi \cap \overset{\bullet}{x} \subseteq \varphi$ now follows $\varphi \in \gamma_\Sigma$.
(4): Let $(\varphi, x) \in q_\Sigma, \overset{\bullet}{y} \supseteq \varphi$. With (1) follows $\varphi \cap \overset{\bullet}{x} \in \gamma_\Sigma$, implying $\varphi \in \gamma_\Sigma$ by proposition 39. Now, $\overset{\bullet}{y} \supseteq \varphi$ yields $\varphi \cap \overset{\bullet}{y} = \varphi$, and so $\varphi \cap \overset{\bullet}{y} \in \gamma_\Sigma$, implying $(\varphi, y) \in q_\Sigma$ by (1).
(5): For each family Σ of coverings on X, by (4) (X, q_Σ) is a R_0–space. If for a topology τ there exist Σ such that $q_\Sigma = q_\tau$, then consequently (X, τ) is R_0, too. If otherwise (X, τ) is R_0, then we take for Σ the family of all open coverings of X and find now trivially $q_\tau \subseteq q_\Sigma$. Let now $(\varphi, x) \in q_\Sigma$. For $U \in \overset{\bullet}{x} \cap \tau$ exists a $V \in \tau$ with $x \notin V \wedge V \supseteq U^c$, by proposition 34. So $\{U, V\} \in \Sigma$ and by definition of q_Σ we have necessary $U \in \varphi$. This leads to $\underline{U}(x) \subseteq \varphi$, implying $(\varphi, x) \in q_\tau$. So, we have $q_\Sigma \subseteq q_\tau$, too. ■

We will come back to this kind of Cauchy filters and convergence with richer structures.

2 Powerfilter–Structures

We are interested to get some nice categorical properties of the generalized uniform structures to define here, like the existence of natural function spaces. In order to realize this, we will - virtually started at uniform structures in the sense of Tukey - enrich these structures, having in mind not only *one* family of sets of subsets (especially coverings), but a set of such families, where furthermore these families are not required to cover the entire base set with each of their members.

2.1 Foundations

2.1.1 Powerfilter-Spaces

42 Definition
Let X be a set and $\mathcal{M} \subseteq \mathfrak{F}(\mathfrak{P}_0(X))$. Then the ordered pair $(X, \mathcal{M})$ is called a **powerfilter-space**, *iff*

(1) $\forall x \in X : \ddot{x} \in \mathcal{M}$ *and*

(2) $\forall \Phi \in \mathcal{M}, \Psi \in \mathfrak{F}(\mathfrak{P}_0(X)) : \Psi \supseteq \Phi \Longrightarrow \Psi \in \mathcal{M}$.

Then $\mathcal{M}$ is called a **powerfilter-structure** *on X.*
If $(X, \mathcal{M}), (Y, \mathcal{N})$ are powerfilter-spaces and $f : X \to Y$ is a mapping, then f is called **fine**, *iff*

(3) $f(\mathcal{M}) := \{[f(\Phi)]_{\mathfrak{F}(\mathfrak{P}_0(Y))} |\ \Phi \in \mathcal{M}\} \subseteq \mathcal{N}$ *holds.*

43 Proposition
The powerfilter-spaces as objects form with the fine maps as morphisms, the usual composition of maps as composition of morphisms and the identical maps as identical morphisms a topological category. The initial powerfilter-structure on a set X w.r.t. $((X_i, \mathcal{M}_i), f_i : X \to X_i)_{i \in I}$ is $\mathcal{M} := \{\Phi \in \mathfrak{F}(\mathfrak{P}_0(X)) |\ \forall i \in I : f_i(\Phi) \in \mathcal{M}_i\}$.

Proof: It's obvious, that the requirements to be a concrete category are fulfilled, so it remains only to show, that the conditions 28(1)-(3) are valid.
fibre-smallness: For a set X, every powerfilter-structure on X is an element of $\mathfrak{P}(\mathfrak{F}(\mathfrak{P}_0(X)))$, thus the class of all powerfilter-structures on X is a subclass of the set $\mathfrak{P}(\mathfrak{F}(\mathfrak{P}_0(X)))$.
terminal separator property: For each singleton $X = \{x\}$ we have $\mathfrak{F}(\mathfrak{P}_0(X)) = \{\ddot{x}\}$ and every powerfilter-structure on X must contain $\ddot{x}$, thus the only powerfilter-structure is $\mathfrak{F}(\mathfrak{P}_0(X))$ itself. For the empty set $\emptyset$ we have $\mathfrak{F}(\mathfrak{P}_0(\emptyset)) = \emptyset$, thus $\mathcal{M} := \emptyset$ is the only powerfilter-structure on $\emptyset$.
initial completeness: Let X be a set and $((X_i, \mathcal{M}_i), f_i : X \to X_i)_{i \in I}$ a indicated class of powerfilter-spaces and mappings from X to their underlying sets. Then $\mathcal{M} := \{\Phi \in \mathfrak{F}(\mathfrak{P}_0(X)) |\ \forall i \in I : f_i(\Phi) \in \mathcal{M}_i\}$ is a powerfilter-structure on X. (Because the image of a singleton-powerfilter is always a singleton-powerfilter and each

mapping preserves inclusion between filters.) Now, each f_i is fine w.r.t. $\mathcal{M}, \mathcal{M}_i$, just by construction of $\mathcal{M}$. Thus, the composition of each f_i with an arbitrary fine map from a powerfilter-space $(Y, \mathcal{N})$ to $(X, \mathcal{M})$ is fine, too. Conversely, let $(Y, \mathcal{N})$ be an arbitrary powerfilter-space and $g : Y \to X$ a function, such that each $f_i \circ g$ is fine w.r.t. $\mathcal{N}, \mathcal{M}_i$. Now, the assumption $g(\mathcal{N}) \not\subseteq \mathcal{M}$ would lead to $\exists \Psi \in \mathcal{N} : g(\Psi) \notin \mathcal{M}$, just meaning $\exists i \in I : f_i(g(\Psi)) \notin \mathcal{M}_i$, by construction of $\mathcal{M}$, in contradiction to our condition that all $f_i \circ g$ are fine. So, $g(\mathcal{N}) \subseteq \mathcal{M}$ must hold, showing, that such a function g is always fine, and thus $\mathcal{M}$ is an initial sructure w.r.t. $((X_i, \mathcal{M}_i), f_i : X \to X_i)_{i \in I}$. In order to prove uniqueness, we assume $\mathcal{M}'$ to be an initial structure w.r.t. these data, too. Then all $f_i \circ \mathbb{1}_X = f_i$ are fine w.r.t. $\mathcal{M}, \mathcal{M}_i$, as seen above, implying $\mathcal{M} \subseteq \mathcal{M}'$ by the initality of $\mathcal{M}'$. But each map $f_i, i \in I$ is fine w.r.t. $\mathcal{M}', \mathcal{M}_i$, too, because of the initial property of $\mathcal{M}'$ (see [43], 1.1.3(1)), thus the same arguments yield $\mathcal{M}' \subseteq \mathcal{M}$. So, $\mathcal{M} = \mathcal{M}'$, implying the uniqueness of the initial structure. ■

We denote the category of powerfilter-spaces and fine maps by **PFS**.

44 Proposition
Let X be a set and $((X_i, \mathcal{M}_i), f_i : X_i \to X)_{i \in I}$ an indicated class of powerfilter-spaces and mappings from their underlying sets to X. Then

$$\mathcal{M} := \{\Phi \in \mathfrak{F}(\mathfrak{P}_0(X))|\ \exists i \in I, \Phi_i \in \mathcal{M}_i : \Phi \supseteq f_i(\Phi_i)\} \cup \{\ddot{x}|\ x \in X\}$$

is the final powerfilter-structure on X w.r.t. $((X_i, \mathcal{M}_i), f_i : X_i \to X)_{i \in I}$.

Proof: By construction, all $f_i, i \in I$ are fine w.r.t. $\mathcal{M}$. Suppose now an arbitrary powerfilter-space $(Y, \mathcal{N})$ and a function $g : X \to Y$ such that all $g \circ f_i$ are fine. Furthermore, assume g not to be fine w.r.t. $\mathcal{M}, \mathcal{N}$. Then there exists a $\Phi \in \mathcal{M}$ with $g(\Phi) \notin \mathcal{N}$. Φ can not be a singleton-powerfilter, because their images are singleton-powerfilters again, which all belong to $\mathcal{N}$ by definition 42. Thus, by construction of $\mathcal{M}$, there must be $i \in I, \Phi_i \in \mathcal{M}_i$ such that $\Phi \supseteq f_i(\Phi_i)$, implying $g(\Phi) \supseteq g(f_i(\Phi_i))$, yielding $g(f_i(\Phi_i)) \notin \mathcal{N}$ - in contradiction to our supposed situation. ■

45 Proposition
Let $(X_i, \mathcal{M}_i)_{i \in I}$ be an indicated class of powerfilter-spaces. Then their product in **PFS** *is $(\prod_{i \in I} X_i, \prod_{i \in I} \mathcal{M}_i)$, where $\prod_{i \in I}$ is the cartesian product of the sets and*

$$\prod_{i \in I} \mathcal{M}_i := \{\Phi \in \mathfrak{F}(\mathfrak{P}_0(\prod_{i \in I} X_i))|\ \exists (\Phi_i \in \mathcal{M}_i)_{i \in I} : \Phi \supseteq \prod_{i \in I} \Phi_i\}$$

with

$$\prod_{i \in I} \Phi_i := \left[\left\{ \{A \in \mathfrak{P}_0(\prod_{i \in I} X_i)|\ p_k(A) \in \varphi_k\} \,\middle|\, k \in I, \varphi_k \in \Phi_k \right\} \right] ,$$

where $p_k : \prod_{i \in I} X_i \to X_k$ are the canonical projections.

Proof: By [43], Th.1.2.1.3, we know, that the product is the initial powerfilter-space on $X := \prod_{i\in I} X_i$ w.r.t. the canonical projections $p_k : \prod_{i\in I} X_i \to X_k$, i.e. it's powerfilter-structure is $\mathcal{M}' := \{\Phi \in \mathfrak{F}(\mathfrak{P}_0(\prod_{i\in I} X_i))|\ \forall i \in I : p_i(\Phi) \in \mathcal{M}_i\}$.
For $(\Phi_i \in \mathcal{M}_i)_{i\in I}$ and any special $k \in I$ the product $\prod_{i\in I} \Phi_i$ contains $\Phi'_k := \{\{A \in \mathfrak{P}_0(\prod_{i\in I} X_i)|\ p_k(A) \in \varphi_k\}|\ \varphi_k \in \Phi_k\}$ as a subset by definition, and obviously $p_k(\Phi'_k) = \Phi_k \in \mathcal{M}_k$ holds, implying $p_k(\prod_{i\in I} \Phi_i) \in \mathcal{M}_k$ by $p_k(\prod_{i\in I} \Phi_i) \supseteq p_k(\Phi'_k)$ for all $k \in I$. (Indeed, $p_k(\prod_{i\in I} \Phi_i) = \Phi_k$ holds.) So, $\prod_{i\in I} \mathcal{M}_i \subseteq \mathcal{M}'$ follows.
Otherwise, for $\Phi \in \mathcal{M}'$ we have $\forall k \in I : \Phi_k := p_k(\Phi) \in \mathcal{M}_k$ by construction of $\mathcal{M}'$ and thus $p_k^{-1}(\Phi_k) \subseteq \Phi$ by proposition 2. But $p_k^{-1}(\Phi_k) = \Phi'_k$ as defined above, so Φ contains all Φ'_k for a certain collection $(\Phi_k \in \mathcal{M}_k)_{k\in I}$, thus it contains the product $\prod_{k\in I} \Phi_k$ and consequently it is contained in $\mathcal{M}$, yielding $\mathcal{M}' \subseteq \mathcal{M}$. ■

46 Proposition
Let $(X, \mathcal{M}) \in |\,\mathbf{PFS}\,|$ and $Y \subseteq X$. Then $\mathcal{M}_{|Y} := \mathcal{M} \cap \mathfrak{F}(\mathfrak{P}_0(Y))$ is the canonical **PFS***–subspace-structure on Y w.r.t. $(X, \mathcal{M})$.*

Proof: Follows immediately from the description of initial structures given in proposition 43. ■

47 Definition
Let $\mathbf{X} := (X, \mathcal{M}), \mathbf{Y} := (Y, \mathcal{N})$ be powerfilter-spaces. We define

$$\mathcal{M}_{\mathbf{X},\mathbf{Y}} := \{\Gamma \in \mathfrak{F}(\mathfrak{P}_0(\mathbf{Y}^{\mathbf{X}}))|\ \forall \Phi \in \mathcal{M} : \omega(\Phi \times \Gamma) \in \mathcal{N}\}$$

with $\Phi \times \Gamma$ defined as in 45.

It is clear, that the singleton-powerfilters on $\mathbf{Y}^{\mathbf{X}}$ all belong to $\mathcal{M}_{\mathbf{X},\mathbf{Y}}$ (because $\mathbf{Y}^{\mathbf{X}}$ contains just the fine maps and $\omega(\Phi \times \ddot{f}) = f(\Phi)$) and that $\mathcal{M}_{\mathbf{X},\mathbf{Y}}$ is closed w.r.t. refinement of powerfilters. Thus $\mathcal{M}_{\mathbf{X},\mathbf{Y}}$ is a powerfilter-structure on $\mathbf{Y}^{\mathbf{X}}$.

48 Proposition
PFS *is a strong topological universe. The natural functionspace-structure on $\mathbf{Y}^{\mathbf{X}}$ for powerfilter-spaces $\mathbf{X} := (X, \mathcal{M}), \mathbf{Y} := (Y, \mathcal{N})$ is $\mathcal{M}_{\mathbf{X},\mathbf{Y}}$. The one-point-extension of a powerfilter-space $(Y, \mathcal{N})$ is $(Y^*, \mathcal{N}^*)$ with $Y^* := Y \cup \{\infty_Y\}, \infty_Y \notin Y$ and $\mathcal{N}^* := \{\Phi \in \mathfrak{F}(\mathfrak{P}_0(Y^*))|\ \mathfrak{F}(\Phi) \cap \mathfrak{F}(\mathfrak{P}_0(Y)) \subseteq \mathcal{N}\}$.*

Proof: For cartesian closedness, by [43], Th.4.1.4, it remains to show, that for any pair $\mathbf{X} := (X, \mathcal{M}), \mathbf{Y} := (Y, \mathcal{N})$ of powerfilter-spaces hold

(1) $\omega : \mathbf{X} \times \mathbf{Y}^{\mathbf{X}} \to \mathbf{Y} : (x, f) \to f(x)$ is fine w.r.t. $\mathcal{M} \times \mathcal{M}_{\mathbf{X},\mathbf{Y}}, \mathcal{N}$ and

(2) $\forall \mathbf{Z} := (Z, \mathcal{O}) \in |\,\mathbf{PFS}\,| : \psi : (\mathbf{Y}^{\mathbf{X}})^{\mathbf{Z}} \to \mathbf{Y}^{\mathbf{X}\times\mathbf{Z}} : g \to \omega \circ (\mathbb{1}_X \times g)$ is surjective.

(1) is clearly fulfilled just by construction of $\mathcal{M}_{\mathbf{X},\mathbf{Y}}$.
(2): Given a powerfilter-space $\mathbf{Z} := (Z, \mathcal{O})$ and an arbitrary fine function $f : X \times Z \to Y$, we define $\overline{f} : Z \to Y^X$ by $\overline{f}(z)(x) := f((x,z))$. Now, we want to show, that $\overline{f}$ is fine, i.e. $\forall \Xi \in \mathcal{O} : \overline{f}(\Xi) \in \mathcal{M}_{\mathbf{X},\mathbf{Y}}$, which is equivalent to $\forall \Phi \in \mathcal{M} : \omega(\Phi \times \overline{f}(\Xi)) \in \mathcal{N}$.
For $\Phi \in \mathcal{M}, \Xi \in \mathcal{O}$ we find $\Phi \times \overline{f}(\Xi) = [\{\{T \in \mathfrak{P}_0(X \times \mathbf{Y}^{\mathbf{X}})|\ p_X(T) \in \varphi \wedge p_{\mathbf{Y}^{\mathbf{X}}}(T) \in \overline{f}(\xi)\}|\ \xi \in \Xi, \varphi \in \Phi\}]$, and for each $\sigma_{\varphi,\xi} := \{T \in \mathfrak{P}_0(X \times \mathbf{Y}^{\mathbf{X}})|\ p_X(T) \in \varphi \wedge p_{\mathbf{Y}^{\mathbf{X}}}(T) \in \overline{f}(\xi)\} \in \Phi \times \overline{f}(\Xi)$ from $p_X(\sigma_{\varphi,\xi}) = \varphi \in \Phi$ and $p_{\mathbf{Y}^{\mathbf{X}}}(\sigma_{\varphi,\xi}) = \overline{f}(\xi)$ it follows, that $\forall T \in \sigma_{\varphi,\xi} : \exists S_T \in \xi, P_T \in \varphi : p_X(P_T) = T \wedge p_{\mathbf{Y}^{\mathbf{X}}}(T) = \overline{f}(S_T)$, so we can build $S'_T := \bigcup_{(x,g)\in T}\{x\} \times (\overline{f}^{-1}(g) \cap S_T) \subseteq X \times Z$ and find $p_X(S'_T) = p_X(T) = P_T, p_Z(S'_T) = S_T$, leading to $\sigma'_{\varphi,\xi} := \{S'_T|\ T \in \sigma_{\varphi,\xi}\}$ with $p_X(\sigma')_{\varphi,\xi} \subseteq \varphi, p_{\mathbf{Y}^{\mathbf{X}}}(\sigma'_{\varphi,\xi}) \subseteq \xi$ and consequently $[\{\sigma'_{\varphi,\xi}|\ \sigma_{\varphi,\xi} \in \Phi \times \overline{f}(\Xi)\}] \supseteq \Phi \times \Xi$. Furthermore, we have $\forall T \in \sigma_{\varphi,\xi} \in \Phi \times \overline{f}(\Xi) : y \in \omega(T) \Leftrightarrow \exists (x,g) \in T : y = g(x) \Leftrightarrow \exists (x, \overline{f}(z)) \in T : f(x,z) = y \Leftrightarrow \exists (x,z) \in S'_T : f(x,z) = y$, yielding $\omega(T) = f(S_T)$ and consequently $\omega(\sigma_{\varphi,\xi}) \subseteq f(\sigma'_{\varphi,\xi})$, thus $\omega(\Phi \times \overline{f}(\Xi)) \supseteq f(\Phi \times \Xi) \in \mathcal{N}$, because f is fine by assumption. And obviously, $\overline{f}$ is a pre-image of f w.r.t. the mapping ψ in condition (2) by construction. Thus ψ is surjective.
For extensionality, we see at first for any powerfilter-space $(Y, \mathcal{N})$, $\Phi \in (\mathcal{N}^*)_{|Y} \Rightarrow \Phi \in \mathcal{N}^* \Leftrightarrow \mathfrak{F}(\Phi) \cap \mathfrak{F}(\mathfrak{P}_0(Y)) \subseteq \mathcal{N} \Rightarrow \Phi \in \mathcal{N}$ and $\Phi \in \mathcal{N} \Rightarrow \Phi \in \mathcal{N}^*$ (because $\Phi \in \mathfrak{F}(\Phi) \cap \mathfrak{F}(\mathfrak{P}_0(Y))) \Rightarrow \Phi \in (\mathcal{N}^*)_{|Y}$ (because $\Phi \in \mathfrak{F}(\mathfrak{P}_0(Y))$), implying $\mathcal{N} = (\mathcal{N}^*)_{|Y}$. Thus $(Y, \mathcal{N})$ is embedded as a subspace in $(Y^*, \mathcal{N}^*)$.
Now, suppose powerfilter-spaces $(X, \mathcal{M}), (Y, \mathcal{N})$, $Z \subseteq X$ and a map $f : Z \to Y$ which is fine w.r.t. $\mathcal{M}_{|Z}, \mathcal{N}$. We have to show, that the map

$$f^* : X \to Y^* :\ f^*(x) := \begin{cases} f(x) & ; \quad \text{if } x \in Z \\ \infty_Y & ; \quad \text{if } x \notin Z \end{cases}$$

is fine w.r.t. $\mathcal{M}, \mathcal{N}^*$. For any $\Phi \in \mathcal{M}$ either $\mathfrak{P}_0(X) \setminus \mathfrak{P}_0(Z) \in \Phi$ holds or $[\Phi \cup \{\mathfrak{P}_0(Z)\}]$ is a proper filter.
In case, that $\mathfrak{P}_0(X) \setminus \mathfrak{P}_0(Z) \in \Phi$ holds, we find $\overset{\bullet}{\infty}_Y \in f^*(\Phi)$, implying $f^*(\Phi) \in \mathcal{N}^*$, because $\overset{\bullet}{\infty}_Y \cap \mathfrak{P}_0(Y) = \emptyset$ and thus $\mathfrak{F}(\Phi) \cap \mathfrak{F}(\mathfrak{P}_0(Y)) = \emptyset \subseteq \mathcal{N}$.
If otherwise $\Phi' := [\Phi \cup \{\mathfrak{P}_0(Z)\}]$ is a proper filter, then it belongs to $\mathcal{M}_{|Z}$ and we have $f^*(\Phi) \supseteq f(\Phi') \cap [\overset{\bullet}{\infty}_Y]_{\mathfrak{F}(\mathfrak{P}_0(Y^*))}$, implying[4] $\forall \Psi \in \mathfrak{F}(f^*(\Phi)) \cap \mathfrak{F}(\mathfrak{P}_0(Y)) : \Psi \supseteq f(\Phi') \in \mathcal{N}$ (because $\Phi' \in M_{|Z}$ and f is fine). So, $\mathfrak{F}(f^*(\Phi)) \cap \mathfrak{F}(\mathfrak{P}_0(Y)) \subseteq \mathcal{N}$ follows and therefore $f^*(\Phi) \in \mathcal{N}^*$ by definition of $\mathcal{N}^*$. Thus $f^*(\mathcal{M}) \subseteq \mathcal{N}^*$.
Products of quotients: Let $(X_i, \mathcal{M}_i), (Y_i, \mathcal{N}_i), i \in I$ be powerfilter-spaces and $f_i : X_i \to Y_i, i \in I$ quotient maps (i.e., all f_i are surjective and each $(Y_i, \mathcal{N}_i)$ is final w.r.t. $((X_i, \mathcal{M}_i), f_i : X_i \to Y_i)$). Now, the map $\prod_{i\in I} f_i : \prod_{i\in I} X_i \to \prod_{i\in I} Y_i : [\prod_{i\in I} f_i]((x_i)_{i\in I}) := (f_i(x_i))_{i\in I}$ is obviously surjective, because all f_i are.

[4]Here $[\overset{\bullet}{\infty}_Y]$ is indicated with $\mathfrak{F}(\mathfrak{P}_0(Y^*))$ to emphasize, that we mean the *principal* filter on $\mathfrak{P}_0(Y^*)$, which is generated by the subset $\overset{\bullet}{\infty}_Y$ of $\mathfrak{P}_0(Y^*)$.

The product $(\prod_{i\in I} Y_i, \prod_{i\in I} \mathcal{N}_i)$ is initial w.r.t. the canonical projections $q_j : \prod_{i\in I} Y_i \to Y_j$, so $\prod_{i\in I} f_i$ is fine, if all composite maps $q_j \circ \prod_{i\in I} f_i$ are fine. But we have naturally $q_j \circ \prod_{i\in I} f_i = f_j \circ p_j$, where the canonical projection $p_j : \prod_{i\in I} X_i \to X_j$ is fine by initiality of the product of the $(X_i, \mathcal{M}_i), i \in I$ and f_j is fine as a quotient map. So, $[\prod_{i\in I} f_i](\prod_{i\in I} \mathcal{M}_i) \subseteq \prod_{i\in I} \mathcal{N}_i$ holds. Otherwise, $\Xi \in \prod_{i\in I} \mathcal{N}_i$ means $\Xi \supseteq \prod_{i\in I} \Xi_i$ for a collection of $\Xi_i \in \mathcal{N}_i, i \in I$ by proposition 45, implying $\Xi \supseteq \prod_{i\in I} f_i(\Sigma_i)$ for a collection of $\Sigma_i \in \mathcal{M}_i, i \in I$ by proposition 44 and finality of the $f_i, i \in I$. By proposition 20 and surjectivity of all $f_i, i \in I$ we get $\Xi \supseteq (\prod_{i\in I} f_i)(\prod_{i\in I} \Sigma_i) \in (\prod_{i\in I} f_i)(\prod_{i\in I} \mathcal{M}_i)$, which yields $\prod_{i\in I} \mathcal{N}_i \subseteq \prod_{i\in I} \mathcal{M}_i$. Thus, by proposition 44, $(\prod_{i\in I} Y_i, \prod_{i\in I} \mathcal{N}_i)$ is final w.r.t. the surjective map $\prod_{i\in I} f_i$ and consequently this is a quotient. ■

49 Definition
A powerfilter-space $(X, \mathcal{M})$ is called **pseudoprincipal**, *iff*

(1) $\forall \Phi \in \mathfrak{F}(\mathfrak{P}_0(X)) : \Phi \in \mathcal{M} \Leftrightarrow \mathfrak{F}_0(\Phi) \subseteq \mathcal{M}$.

A powerfilter-space $(X, \mathcal{M})$ is called **refinement-closed**, *iff*

(2) $\forall \Phi \in \mathcal{M}, \Psi \in \mathfrak{F}(\mathfrak{P}_0(X)) : \Psi \preceq \Phi \Rightarrow \Psi \in \mathcal{M}$.

It is clear, that the pseudoprincipal powerfilter-spaces form a full and isomorphism-closed subcategory of **PFS**. We denote it by **psPFS**. The refinement-closed powerfilter-spaces form a full and isomorphism-closed subcategory of **PFS**, too, as is easy to see. We denote it by $\mathbf{PFS}^{\preceq}$.

50 Proposition
$\mathbf{PFS}^{\preceq}$ *is a bireflective subcategory of* **PFS**.

Proof: For a powerfilter-space $(X, \mathcal{M})$ we define the corresponding refinement-closed powerfilterspace as $(X, \mathcal{M}^{\preceq})$ by

$$\mathcal{M}^{\preceq} := \{\Psi \in \mathfrak{F}(\mathfrak{P}_0(X)) |\ \exists \Phi \in \mathcal{M} : \Psi \preceq \Phi\}\ .$$

Then $\mathcal{M} \subseteq \mathcal{M}^{\preceq}$ follows from the reflexivity of the $\preceq$-relation, for every powerfilter-structure $\mathcal{M}$ holds $(\mathcal{M}^{\preceq})^{\preceq} = \mathcal{M}^{\preceq}$ and a powerfilter-structure $\mathcal{N}$ is refinement-closed, if and only if $\mathcal{N} = \mathcal{N}^{\preceq}$. Thus, $\mathbb{1}_X : (X, \mathcal{M}) \to (X, \mathcal{M}^{\preceq})$ is fine and if for any $(Y, \mathcal{N}) \in |\,\mathbf{PFS}^{\preceq}\,|$ a map $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is fine, then we have by definition $f(\mathcal{M}) \subseteq \mathcal{N}$, implying $f(\mathcal{M}^{\preceq}) \subseteq f(\mathcal{M})^{\preceq} \subseteq \mathcal{N}^{\preceq} = \mathcal{N}$. ■

2.1.2 Multifilter-Spaces

51 Definition
For a set X and a set $\mathcal{M}$ of multifilters on X we call the pair $(X, \mathcal{M})$ a **multifilter-space**, *iff*

(1) $\forall x \in X : \widehat{x} \in \mathcal{M}$ *and*

(2) $\Sigma_1 \in \mathcal{M} \wedge \Sigma_2 \preceq \Sigma_1 \Rightarrow \Sigma_2 \in \mathcal{M}$

holds. $\mathcal{M}$ *is called the* **multifilter-structure** *of this space.*
If $(X_1, \mathcal{M}_1)$, $(X_2, \mathcal{M}_2)$ *are multifilter-spaces and* $f : X_1 \to X_2$ *is a map, then* f *is called* **fine** *(w.r.t.* $\mathcal{M}_1, \mathcal{M}_2$*), iff*

(3) $f(\mathcal{M}_1) \subseteq \mathcal{M}_2$.

A multifilter-space $(X, \mathcal{M})$ *is called* **limited**, *iff*

(4) $\forall \Sigma_1, \Sigma_2 \in \mathcal{M} : \Sigma_1 \cap \Sigma_2 \in \mathcal{M}$,

it is called **principal**, *iff*

(5) $\exists \Sigma_0 \in \mathcal{M} : \forall \Sigma \in \mathcal{M} : \Sigma \preceq \Sigma_0$.

A limited multifilter-space $(X, \mathcal{M})$ *is called a* **weakly uniform** *limited multifilter-space, iff*

(6) $\forall \Sigma \in \mathcal{M} : \Sigma^\diamond \in \mathcal{M}$ *with* $\Sigma^\diamond := \{\alpha \in \mathfrak{P}_0(\mathfrak{P}_0(X))|\ \exists \sigma \in \Sigma : \sigma^\diamond \preceq \alpha\}$ *and* $\sigma^\diamond := \{\bigcup_{i=1}^n S_i|\ n \in I\!N, S_i \in \sigma, \exists x \in X : \forall i = 1, ...n : x \in S_i, \bigcup_{i=1}^n S_i \neq \emptyset\}$.

it is called a **uniform** *limited multifilter-space, iff*

(7) $\forall \Sigma \in \mathcal{M} : \Sigma^* \in \mathcal{M}$ *with* $\Sigma^* := \{\alpha \in \mathfrak{P}_0(\mathfrak{P}_0(X))|\ \exists \sigma \in \Sigma : \sigma^* \preceq \alpha\}$ *and* $\sigma^* := \{st(x, \sigma)|\ x \in X, st(x, \sigma) \neq \emptyset\}$.

Every multifilter or powerfilter which refines a member of $\mathcal{M}$ is called *fine* (w.r.t. the multifilter-structure $\mathcal{M}$).

Note, that every uniform limited multifilter-space is weakly uniform, which becomes immediately clear from the fact, that $\sigma^\diamond \preceq \sigma^*$ for every partial cover σ of a set X.

52 Proposition
The multifilter-spaces as objects and the fine mappings between them as morphisms form a topological category with the usual composition of mappings as composition of morphisms and the identical functions as identical morphisms.

Proof: It's obvious, that the requirements to form a category are fulfilled, and this category is concrete by construction. So we have only to show, that this category is topological.
For any set X, all multifilter-structures on X are elements of $\mathfrak{P}(\mathfrak{P}_0(\mathfrak{P}_0(\mathfrak{P}_0(X))))$, so the class of all multifilter-structures on X is a subclass of $\mathfrak{P}(\mathfrak{P}_0(\mathfrak{P}_0(\mathfrak{P}_0(X))))$ and therefore it is a set, too.
For any singleton $X := \{x\}$, the only multifilter on X is $\widehat{x}$, which must be contained in each multifilter-structure on X by definition. On the empty set $X := \emptyset$ the empty

structure $\mathcal{M} := \emptyset$ is indeed a multifilter-structure and it's the only one. So, on any set X of cardinality at most one there exists precisely one multifilter-structure.
For any set X, any family $((X_i, \mathcal{M}_i))_{i\in I}$ of multifilter-spaces indicated by a class I and any family $(f_i : X \to X_i)_{i\in I}$ indicated by the same class I we can define $\mathcal{M} := \{\Sigma \in \widehat{\mathfrak{F}}(X) | \forall i \in I : f_i(\Sigma) \in \mathcal{M}_i\}$, which is obviously a multifilter-structure on X, because of proposition 21(1) and the fact, that an image of a singleton-multifilter is always a singleton-multifilter. By construction, for each $i \in I$, f_i is a fine map w.r.t. $(\mathcal{M}, \mathcal{M}_i)$. So, for any multifilter-space $(Y, \mathcal{N})$ and any fine map $g : Y \to X$ (w.r.t. $(\mathcal{N}, \mathcal{M})$) it follows, that the composite maps $f_i \circ g$ are fine. Conversely, for a given multifilter-space $(Y, \mathcal{N})$ and a map $g : Y \to X$, whose composites $f_i \circ g$ are fine for all $i \in I$, the map g itself must be fine: assuming the contrary, it would follow, that there exists a multifilter $\mathrm{T} \in \mathcal{N}$ with $g(\mathrm{T}) \notin \mathcal{M}$, which implies $\exists i_0 \in I : f_{i_0}(g(\mathrm{T})) \notin \mathcal{M}_{i_0}$ because of the special construction of $\mathcal{M}$, and so the composite map $f_{i_0} \circ g$ would not be fine. Therefore, $\mathcal{M}$ is initial w.r.t. $(X, (f_i, X_i, \mathcal{M}_i)_{i\in I})$. Assume, there is an initial structure $\mathcal{M}'$ w.r.t. $(X, (f_i, X_i, \mathcal{M}_i)_{i\in I})$. Then the identical map $\mathbb{1}_X$ is fine w.r.t. $(\mathcal{M}, \mathcal{M}')$, because of the initial property of $\mathcal{M}'$ and the fact, that all composites $f_i \circ \mathbb{1}_X$ are fine, yielding $\mathcal{M} \subseteq \mathcal{M}'$. But each map $f_i, i \in I$ is fine w.r.t. $(\mathcal{M}', \mathcal{M}_i)$, too, because of the initial property of $\mathcal{M}'$ (see [43]), and now the same argument yields $\mathcal{M}' \subseteq \mathcal{M}$. So, $\mathcal{M}$ is the unique initial multifilter-structure on X w.r.t. $(X, (f_i, X_i, \mathcal{M}_i)_{i\in I})$. ■

We denote the category of multifilter-spaces and fine maps by **MFS**.

The (obviously full and isomorphism-closed) subcategories of limited, principal, weak uniform limited, weak uniform principal, uniform limited and uniform principal multifilter-spaces are denoted by **LimMFS**, **PrMFS**, **WULimMFS**, **PrWULimMFS**, **ULimMFS** and **PrULimMFS**, respectively.

53 Lemma

(1) **LimMFS** *is bireflective in* **MFS**.

(2) **PrMFS** *is bireflective in* **LimMFS**.

(3) **ULimMFS** *is bireflective in* **LimMFS**.

(4) **WULimMFS** *is bireflective in* **LimMFS**.

(5) **PrULimMFS** *is bireflective in* **LimMFS**.

(6) **PrWULimMFS** *is bireflective in* **LimMFS**.

Proof: (1): Let $(X, \mathcal{M}) \in |\,\mathbf{MFS}\,|$ and let
$\mathcal{M}^{lim} := \{\Sigma \in \widehat{\mathfrak{F}}(X) |\ \exists n \in I\!N, \Sigma_1, ..., \Sigma_n \in \mathcal{M} : \Sigma \preceq \bigcap_{i=1}^{n} \Sigma_i\}$, which is naturally a limited multifilter-structure on X, trivially refined by $\mathcal{M}$, thus $\mathbb{1}_X : (X, \mathcal{M}) \to (X, \mathcal{M}^{lim})$ is fine. For each $(Y, \mathcal{N}) \in |\,\mathbf{LimMFS}\,|$ and $f \in [X, Y]_{\mathbf{MFS}}$, we get from

$f(\mathcal{M}) \subseteq \mathcal{N}$ easily $f(\mathcal{M}^{lim}) = f(\mathcal{M})^{lim} \subseteq \mathcal{N}^{lim} = \mathcal{N}$ by proposition 17(3).
(2): Let $(X, \mathcal{M}) \in |\,\mathbf{LimMFS}\,|$ and let $\mathcal{M}^{prlim} := \{\Xi \in \widehat{\mathfrak{F}}(X) |\ \Xi \preceq \bigcap_{\Sigma \in \mathcal{M}} \Sigma\}$, which is naturally a principal multifilter-structure on X, trivially refined by $\mathcal{M}$, thus $\mathbb{1}_X : (X, \mathcal{M}) \to (X, \mathcal{M}^{prlim})$ is fine. For each $(Y, \mathcal{N}) \in |\,\mathbf{PrMFS}\,|$ and $f \in [X, Y]_{\mathbf{MFS}}$, we get from $f(\mathcal{M}) \subseteq \mathcal{N}$ now $f(\mathcal{M}^{prlim}) = f(\mathcal{M})^{prlim} \subseteq \mathcal{N}^{prlim} = \mathcal{N}$ by proposition 17(3), again.
(3): Let $(X, \mathcal{M}) \in |\,\mathbf{LimMFS}\,|$ and let $\mathcal{M}^{ulim} := \{\Xi \in \widehat{\mathfrak{F}}(X) |\ \exists \Sigma \in \mathcal{M} : \Xi \preceq \Sigma^{*n}\}$, where Σ^{*n} is the multifilter derived by applying n-times the *-operator to Σ. $\mathcal{M}^{ulim}$ is refined by $\mathcal{M}$, because of proposition 17(5), thus $\mathbb{1}_X : (X, \mathcal{M}) \to (X, \mathcal{M}^{prlim})$ is fine. $\mathcal{M}^{ulim}$ is a limited multifilter-structure on X again, because easily $\Sigma_1^{*n} \cap \Sigma_2^{*m} \preceq (\Sigma_1 \cap \Sigma_2)^{*(n+m)}$ follows from proposition 17(6)(5)(2). Moreover, $\mathcal{M}^{ulim}$ is obviously uniform, by construction. For any $(Y, \mathcal{N}) \in |\,\mathbf{ULimMFS}\,|$ and $f \in [X, Y]_{\mathbf{MFS}}$, we get from $f(\mathcal{M}) \subseteq \mathcal{N}$ now $f(\mathcal{M}^{ulim}) \subseteq f(\mathcal{M})^{ulim} \subseteq \mathcal{N}^{ulim} = \mathcal{N}$, because of proposition 17(4).
(4): Let $(X, \mathcal{M}) \in |\,\mathbf{LimMFS}\,|$ and let $\mathcal{M}^{wulim} := \{\Xi \in \widehat{\mathfrak{F}}(X) |\ \exists \Sigma \in \mathcal{M}, n \in I\!N : \Xi \preceq \Sigma^{\diamond n}\}$, where $\Sigma^{\diamond n}$ is the multifilter derived by applying n times the $^\diamond$-operator. Now, all things work similar to the foregoing case, with weak uniform instead of uniform, $\diamond$ instead of $*$ and $\mathcal{M}^{wulim}$ instead of $\mathcal{M}^{ulim}$.
(5): Follows from (3) and (2), because intersections of bireflective subcategories of a topological category are bireflective, too.
(6): Follows from (4) and (2). ■

54 Lemma
The category **UMer** *of uniform covering spaces (in the sense of Tukey) and uniformly continuous maps is concretely isomorphic to* **PrULimMFS**.

Proof: If $(X, \mathcal{M})$ is a principal uniform multifilter-space, then $\mathcal{M} = [\Sigma_0] := \{\Sigma \in \widehat{\mathfrak{F}}(X) |\ \Sigma \preceq \Sigma_0\}$ and $\Sigma_0^* \in \mathcal{M}$, thus $\Sigma_0^* = \Sigma_0$, because of proposition 17(5). So, for each $\alpha \in \Sigma_0$, there is a $\beta \in \Sigma_0$ which is a barycentric refinement of α. For all $\alpha, \beta \in \Sigma_0$ we have $\alpha \wedge \beta \in \Sigma_0$, because Σ is a multifilter. Furthermore, it is clear, that every member of Σ_0 must cover X entirely, because every singleton-multifilter $[\{\{x\}\}]$ refines Σ_0. So, Σ_0 itself is a uniform structure in the sense of Tukey and for a fine map f from a principal uniform multifilter-space $(X, [\Sigma])$ to a principal uniform multifilter-space $(Y, [\Xi])$ fulfills $f(\Sigma) \preceq \Xi$, so it is uniformly continuous between the uniform spaces $(X, \Sigma), (Y, \Xi)$. Thus we have a functor $U : \mathbf{PrULimMFS} \to \mathbf{UMer} : (X, [\Sigma_0]) \to (X, \Sigma_0)$, which works identically on the morphisms-maps.
On the other hand, if (X, Σ_0) is any uniform space, then $(X, [\Sigma_0])$ is clearly a principal uniform multifilter-space, because it is principal by construction and uniform by the star-refinement property of Σ_0. Moreover, for a uniformly continuous map f from a uniform space (X, Σ) to a uniform space (Y, Ξ) always holds $f(\Sigma) \preceq \Xi$, implying $f([\Sigma]) \subseteq [\Xi]$, so it is a fine map between the principal uniform multifilter-

spaces $(X,[\Sigma]),(Y,[\Xi])$. Thus, we have a functor $P : \mathbf{UMer} \to \mathbf{PrULimMFS} : (X,\Sigma) \to (X,[\Sigma])$, which works identically on the morphism-maps.
Finally, $U \circ P = \mathbb{1}_{\mathbf{UMer}}$ and $P \circ U = \mathbb{1}_{\mathbf{PrULimMFS}}$ follow directly from the definitions of these functors. ■

By [43], Th. 1.2.1.1. we know, that arbitrary final structures exist, too, in a topological category.

55 Proposition
Let X be a set, $((X_i,\mathcal{M}_i), f_i : X_i \to X)_{i\in I}$ an indicated class of multifilter-spaces and mappings from them to X. Then

$$\mathcal{M} := \{\Sigma \in \widehat{\mathfrak{F}}(X) |\ \exists i \in I, \Sigma_i \in \mathcal{M}_i : \Sigma \preceq f_i(\Sigma_i)\} \cup \{\widehat{x} |\ x \in X\}$$

is the final multifilter-structure on X w.r.t. $((X_i,\mathcal{M}_i), f_i : X_i \to X)_{i\in I}$.

Proof: By construction, all $f_i, i \in I$ are fine w.r.t. $\mathcal{M}$. Now, suppose an arbitrary multifilter-space $(Y,\mathcal{N})$ and a function $g : X \to Y$ such that all $g \circ f_i$ are fine. Furthermore, suppose g not to be fine w.r.t. $\mathcal{M},\mathcal{N}$. Then there is a $\Sigma \in \mathcal{M}$ with $g(\Sigma) \notin \mathcal{N}$. Σ can not be a singleton-multifilter, because the images of the singleton-multifilters are singleton-multifilters, which naturally all belongs to $\mathcal{N}$. So, by construction of $\mathcal{M}$ there must be a $i \in I, \Sigma_i \in \mathcal{M}_i$ such that $\Sigma \preceq f_i(\Sigma_i)$. But then would follow $g(\Sigma) \preceq g(f_i(\Sigma_i))$, which implies $g(f_i(\Sigma_i)) \notin \mathcal{N}$ – in contradiction to our supposed situation. ■

56 Proposition
Let $(X_i,\mathcal{M}_i)_{i\in I}$ be an indicated class of multifilter-spaces. Then their product in **MFS** *is $(\prod_{i\in I} X_i, \prod_{i\in I}\mathcal{M}_i)$ where $\prod X_i$ is the cartesian product of the sets and $\prod_{i\in I}\mathcal{M}_i$ is the set of all multifilters Σ on $\prod_{i\in I} X_i$ for which there exists $\Sigma_i \in \mathcal{M}_i, i \in I$ such that $\Sigma \preceq \prod_{i\in I}\Sigma_i$.*

Proof: By [43], Th. 1.2.1.3, we know, that the product is the initial multifilter-space on $X := \prod_{i\in I} X_i$ w.r.t. the canonical projections $p_k : \prod_{i\in I} X_i \to X_k$, i.e. it's multifilter-structure is $\mathcal{M}' := \{\Sigma \in \widehat{\mathfrak{F}}(\prod_{i\in I} X_i) |\ \forall i \in I : p_i(\Sigma) \in \mathcal{M}_i\}$.
For $\Sigma_i \in \mathcal{M}_i, i \in I$ and any special $k \in I$ the product $\prod_{i\in I}\Sigma_i$ contains
$\Sigma'_k := \left\{\prod_{i\in I}\sigma_i |\ \sigma_k \in \Sigma_k \wedge \forall i \in I \setminus \{k\} : \sigma_i = \{X_i\}\right\}$ as a subset, by definition, and $p_k(\Sigma'_k) = \Sigma_k \in \mathcal{M}_k$. Now, $\prod_{i\in I}\Sigma_i$ can not be coarser than one of it's subsets, which $p_k(\prod_{i\in I}\Sigma_i) \in \mathcal{M}_k$ implies. (Indeed, $p_k(\prod_{i\in I}\Sigma_i) = \Sigma_k$ holds.) So, all products $p_k(\prod_{i\in I}\Sigma_i)$ with $\Sigma_i \in \mathcal{M}_i, i \in I$ are members of $\mathcal{M}'$, yielding $\prod_{i\in I}\mathcal{M}_i \subseteq \mathcal{M}'$. Otherwise, if Σ belongs to $\mathcal{M}'$, we have $\forall k \in I : p_k(\Sigma) =: \Sigma_k \in \mathcal{M}_k$, implying $p_k^{-1}(\Sigma_k) \subseteq \Sigma$ by proposition 21(2). But $p_k^{-1}(\Sigma_k) = \Sigma'_k$ as defined above. Because this holds for all $k \in I$, Σ contains the product $\prod_{k\in I}\Sigma_k$, i.e. $\Sigma \preceq \prod_{k\in I}\Sigma_k$. This holds for all $\Sigma \in \mathcal{M}'$, so $\mathcal{M}' \subseteq \prod_{i\in I}\mathcal{M}_i$. ■

57 Definition
Let $\mathbf{X}=(X,\mathcal{M}),\mathbf{Y}=(Y,\mathcal{N})\in|\,\mathbf{MFS}\,|$. *We define*

$$\mathcal{M}_{\mathbf{X},\mathbf{Y}}:=\{\Gamma\in\widehat{\mathfrak{F}}(\mathbf{Y}^{\mathbf{X}})|\ \forall\Sigma\in\mathcal{M}:\Gamma(\Sigma)\in\mathcal{N}\}\ .$$

(In the above statement we mean $\Gamma(\Sigma):=[\{\gamma(\sigma)|\ \gamma\in\Gamma,\sigma\in\Sigma\}]$, $\gamma(\sigma):=\{G(S)|\ G\in\gamma,S\in\sigma\}$ *and* $G(S):=\{y\in Y|\ \exists g\in G,s\in S:y=g(s)\}=\omega(S\times G)$ *with the evaluation map* ω*).*

It's obvious, that the singleton-multifilters on $\mathbf{Y}^{\mathbf{X}}$ all belong to $\mathcal{M}_{\mathbf{X},\mathbf{Y}}$ (because $\mathbf{Y}^{\mathbf{X}}$ contains just the fine maps) and that $\mathcal{M}_{\mathbf{X},\mathbf{Y}}$ is closed w.r.t. refinement of multifilters. So, $\mathcal{M}_{\mathbf{X},\mathbf{Y}}$ is a multifilter-structure on $\mathbf{Y}^{\mathbf{X}}$.

58 Lemma
MFS *is a strong topological universe. The natural function-space of the multifilter-spaces* $\mathbf{X}:=(X,\mathcal{M})$ *and* $\mathbf{Y}:=(Y,\mathcal{N})$ *is* $(\mathbf{Y}^{\mathbf{X}},\mathcal{M}_{\mathbf{X},\mathbf{Y}})$.

Proof: *Cartesian closedness.* By [43], Th.4.1.4., it remains to show, that for any pair $\mathbf{X}=(X,\mathcal{M}),\mathbf{Y}=(Y,\mathcal{N})$ of multifilter-spaces hold

(1) $\omega:\mathbf{X}\times\mathbf{Y}^{\mathbf{X}}\to\mathbf{Y}:(x,f)\to f(x)$ is fine w.r.t. $\mathcal{M}\times\mathcal{M}_{\mathbf{X},\mathbf{Y}},\mathcal{N}$ and

(2) $\forall\mathbf{Z}=(Z,\mathcal{O})\in|\,\mathbf{MFS}\,|:\ \psi:(\mathbf{Y}^{\mathbf{X}})^{\mathbf{Z}}\to\mathbf{Y}^{\mathbf{X}\times\mathbf{Z}}:g\to\omega\circ(\mathbb{1}_X\times g)$ is surjective.

(1) For $\mathbf{X}:=(X,\mathcal{M}),\mathbf{Y}:=(Y,\mathcal{N})\in|\,\mathbf{MFS}\,|$, the evaluation map $\omega:\mathbf{X}\times\mathbf{Y}^{\mathbf{X}}\to\mathbf{Y}:\omega(x,f):=f(x)$ is fine, because $\omega(\mathcal{M}\times\mathcal{M}_{\mathbf{X},\mathbf{Y}})=\omega([\{\Sigma\times\Gamma|\ \Sigma\in\mathcal{M},\Gamma\in\mathcal{M}_{\mathbf{X},\mathbf{Y}}\}])=[\{\omega(\Sigma\times\Gamma)|\ \Sigma\in\mathcal{M},\Gamma\in\mathcal{M}_{\mathbf{X},\mathbf{Y}}\}]$ and $\omega(\Sigma\times\Gamma)=\omega([\{\sigma\times\gamma|\ \sigma\in\Sigma,\gamma\in\Gamma\}])=[\{\omega(\sigma\times\gamma)|\sigma\in\Sigma,\gamma\in\Gamma\}]=[\{\{\omega(S\times G)|S\in\sigma,G\in\gamma\}|\ \sigma\in\Sigma,\gamma\in\Gamma\}]=[\{\{G(S)|S\in\sigma,G\in\gamma\}|\ \sigma\in\Sigma,\gamma\in\Gamma\}]\in\mathcal{N}$ for $\Sigma\in\mathcal{M}$ and $\Gamma\in\mathcal{M}_{\mathbf{X},\mathbf{Y}}$, by definition of $\mathcal{M}_{\mathbf{X},\mathbf{Y}}$.
(2) Given any multifilter-space $\mathbf{Z}:=(Z,\mathcal{O})$ and an arbitrary fine function $f:\mathbf{X}\times\mathbf{Z}\to\mathbf{Y}$, we define $\overline{f}:\mathbf{Z}\to(\mathbf{Y}^{\mathbf{X}})$ by $\overline{f}(z)(x):=f((x,z))$. Then we have $\forall O\in\xi\in\Xi\in\mathcal{O},A\in\alpha\in\Sigma\in\mathcal{M}:\overline{f}(O)(A)=\{\overline{f}(z)(a)|z\in O,a\in A\}=\{f(a,z)|a\in A,z\in O\}=f(A\times O)$, implying $\overline{f}(\xi)(\alpha)=\{\overline{f}(O)(A)|O\in\xi,A\in\alpha\}=\{f(A\times O)|A\in\alpha,O\in\xi\}=f(\alpha\times\xi)$ and thus $\overline{f}(\Xi)(\Sigma)=\{\beta\in\mathfrak{P}_0(\mathfrak{P}_0(Y))|\exists\xi\in\Xi,\alpha\in\Sigma:f(\alpha\times\xi)\preceq\beta\}=f(\Sigma\times\Xi)\in\mathcal{N}$, because f is fine w.r.t. $\mathcal{M}\times\mathcal{O},\mathcal{N}$. Now $\overline{f}(\mathcal{O})\subseteq\mathcal{M}_{X,Y}$ follows, so $\overline{f}$ is fine and it is a pre-image of f w.r.t. the mapping ψ in condition (2) by construction.
Extensionality. For $\mathbf{Z}:=(Z,\mathcal{O})\in|\,\mathbf{MFS}\,|$ we define the *one-point-extension* $\mathbf{Z}^*:=(Z^*,\mathcal{O}^*)$ by $Z^*:=Z\cup\{\infty_Z\}$ with a point $\infty_Z\notin Z$ and $\mathcal{O}^*:=\{\Sigma\in\widehat{\mathfrak{F}}(Z^*)|\ \Sigma_{|Z}\in\mathcal{O}\}\cup\{\widehat{\infty_Z}\}$, which clearly fulfills both of the defining conditions of a multifilter-space. Now, given any multifilter-space $(X,\mathcal{M})$, a subset $Y\subseteq X$ and a fine mapping $f:Y\to Z$ w.r.t. $\mathcal{M}_{|Y},\mathcal{O}$, we have to show, that $f:X\to Z^*$, defined by

$$f^*(x):=\begin{cases} f(x) & ;\ x\in Y\\ \infty_Z & ;\ x\notin Y\end{cases}$$

is fine w.r.t. $\mathcal{M}, \mathcal{O}^*$. But, for all $\Sigma \in \mathcal{M}$ we have either that $\Sigma_{|Y}$ exists as a multifilter and so it belongs to $\mathcal{M}_{|Y}$, implying $f^*(\Sigma)_{|Z} = f(\Sigma_{|Y}) \in \mathcal{O}$ because f is fine, or $\Sigma_{|Y}$ doesn't exist, which means $\{\{X \setminus Y\}\} \in \Sigma$, implying $f^*(\Sigma) = \widehat{\infty_Y}$. In each of both cases, $f^*(\Sigma) \in \mathcal{O}^*$ follows.
Products of Quotients. Let $(X_i, \mathcal{M}_i), (Y_i, \mathcal{N}_i)$ be multifilter-spaces and $f_i : X_i \to Y_i$ quotient maps (i.e., all f_i are surjective and each $(Y_i, \mathcal{N}_i)$ is final w.r.t. f_i). Now, the map $\prod_{i\in I} f_i : \prod_{i\in I} X_i \to \prod_{i\in I} Y_i : [\prod_{i\in I} f_i]((x_i)_{i\in I}) := (f_i(x_i))_{i\in I}$ is obviously surjective, because all f_i are.
The product $(\prod_{i\in I} Y_i, \prod_{i\in I} \mathcal{N}_i)$ is initial w.r.t. the canonical projections $q_j : \prod_{i\in I} Y_i \to Y_j$, so $\prod_{i\in I} f_i$ is fine, if all composite maps $q_j \circ \prod_{i\in I} f_i$ are fine. But we have naturally $q_j \circ \prod_{i\in I} f_i = f_j \circ p_j$, where the canonical projection $p_j : \prod_{i\in I} X_i \to X_j$ is fine by initiality of the product of the $(X_i, \mathcal{M}_i), i \in I$ and f_j is fine as a quotient map. So, $[\prod_{i\in I} f_i](\prod_{i\in I} \mathcal{M}_i) \subseteq \prod_{i\in I} \mathcal{N}_i$ holds. Otherwise, $\Xi \in \prod_{i\in I} \mathcal{N}_i$ means $\Xi \preceq \prod_{i\in I} \Xi_i$ for a collection of $\Xi_i \in \mathcal{N}_i, i \in I$ by proposition 56, implying $\Xi \preceq \prod_{i\in I} f_i(\Sigma_i)$ for a collection of $\Sigma_i \in \mathcal{M}_i, i \in I$ by proposition 55 and finality of the $f_i, i \in I$. By proposition 21(3) and surjectivity of all $f_i, i \in I$ we get $\Xi \preceq (\prod_{i\in I} f_i)(\prod_{i\in I} \Sigma_i) \in (\prod_{i\in I} f_i)(\prod_{i\in I} \mathcal{M}_i)$, which yields $\prod_{i\in I} \mathcal{N}_i \subseteq \prod_{i\in I} \mathcal{M}_i$. Thus, by proposition 55, $(\prod_{i\in I} Y_i, \prod_{i\in I} \mathcal{N}_i)$ is final w.r.t. the surjective map $\prod_{i\in I} f_i$ and consequently it is a quotient. ■

59 Lemma
MFS *is concretely isomorphic to* $\mathbf{PFS}^{\preceq}$.

Proof: For $(X, \mathcal{M}) \in |\,\mathbf{MFS}\,|$ let $\mathcal{M}^p := \{\Phi \in \mathfrak{F}(\mathfrak{P}_0(X))|\ \exists \Sigma \in \mathcal{M} : \Phi \preceq \Sigma\}$, which is of course a powerfilter-structure on X and for $(Y, \mathcal{N}) \in |\,\mathbf{PFS}\,|$ let $\mathcal{N}^m := \{\Sigma \in \widehat{\mathfrak{F}}(Y)|\ \exists \Phi \in \mathcal{N} : \Sigma \preceq \Phi\}$, which is obviously a multifilter-structure on Y. Then holds $(\mathcal{M}^p)^m = \{\Sigma \in \widehat{\mathfrak{F}}(X)|\ \exists \Phi \in \mathcal{M}^p : \Sigma \preceq \Phi\} = \{\Sigma \in \widehat{\mathfrak{F}}(X)|\ \exists \Sigma' \in \mathcal{M}, \Phi \in \mathfrak{F}(\mathfrak{P}_0(X)) : \Sigma \preceq \Phi \preceq \Sigma'\} = \mathcal{M}$ by proposition 27(5) and $(\mathcal{N}^m)^p = \{\Phi \in \mathfrak{F}(\mathfrak{P}_0(Y))|\ \exists \Sigma \in \mathcal{N}^m : \Phi \preceq \Sigma\} = \{\Phi \in \mathfrak{F}(\mathfrak{P}_0(Y))|\ \exists \Phi' \in \mathcal{N}, \Sigma \in \widehat{\mathfrak{F}}(Y) : \Phi \preceq \Sigma \preceq \Phi'\} = \mathcal{N}$ by proposition 27(4).
Furthermore, if $(X_1, \mathcal{M}_1), (X_2, \mathcal{M}_2) \in |\,\mathbf{MFS}\,|$ and $f : X_1 \to X_2$ with $f(\mathcal{M}_1) \subseteq \mathcal{M}_2$, then $f(\mathcal{M}_1^p) = \{[f(\Phi)]_{\mathfrak{F}(\mathfrak{P}_0(X_2))}|\ \Phi \in \mathfrak{F}(\mathfrak{P}_0(X_1)), \exists \Sigma \in \mathcal{M}_1 : \Phi \preceq \Sigma\} \subseteq \{\Psi \in \mathfrak{F}(\mathfrak{P}_0(X_2))|\ \exists \Sigma \in \mathcal{M}_1 : \Psi \preceq f(\Sigma)\} \subseteq \{\Psi \in \mathfrak{F}(\mathfrak{P}_0(X_2))|\ \exists \Sigma' \in \mathcal{M}_2 : \Psi \preceq \Sigma'\} = \mathcal{M}_2^p$ and if $(Y_1, \mathcal{N}_1), (Y_2, \mathcal{N}_2) \in |\,\mathbf{PFS}\,|$, $f : Y_1 \to Y_2$ with $f(\mathcal{N}_1) \subseteq \mathcal{N}_2$ are given, then $f(\mathcal{N}_1^m) = \{[f(\Sigma)]_{\widehat{\mathfrak{F}}(Y_2)}|\ \Sigma \in \widehat{\mathfrak{F}}(Y_1), \exists \Phi \in \mathcal{N}_1 : \Sigma \preceq \Phi\} \subseteq \{\Xi \in \widehat{\mathfrak{F}}(Y_2)|\ \exists \Phi \in \mathcal{N}_1 : \Xi \preceq f(\Phi)\} \subseteq \{\Xi \in \widehat{\mathfrak{F}}(Y_2)|\ \exists \Phi' \in \mathcal{N}_2 : \Xi \preceq \Phi'\} = \mathcal{N}_2^m$. So, each fine map f between $(X_1, \mathcal{M}_1), (X_2, \mathcal{M}_2)$ in **MFS** is fine w.r.t. $\mathcal{M}_1^p, \mathcal{M}_2^p$ in **PFS** and each fine map f between $(Y_1, \mathcal{N}_1), (Y_2, \mathcal{N}_2)$ in **PFS** is fine w.r.t. $\mathcal{N}_1^m, \mathcal{N}_2^n$ in **MFS**, too. Thus, $F : \mathbf{MFS} \to \mathbf{PFS} : (X, \mathcal{M}) \to (X, \mathcal{M}^p), f \to f$ and $G : \mathbf{PFS} \to \mathbf{MFS} : (Y, \mathcal{N}) \to (Y, \mathcal{N}^m), f \to f$ are functors, which are obviously concrete, and $F \circ G = \mathbb{1}_{\mathbf{PFS}}, G \circ F = \mathbb{1}_{\mathbf{MFS}}$ hold. ■

60 Corollary
PFS$^{\preceq}$ *is a strong topological universe.*

61 Definition
A multifilter-space $(X, \mathcal{M})$ *is called* **pseudoprincipal**, *iff its corresponding powerfilter-space* $(X, \mathcal{M}^p)$ *is pseudoprincipal.*

62 Proposition
Let $(X, \mathcal{M})$ *be a multifilter-space. Then the following are equivalent:*

(1) $(X, \mathcal{M})$ *is pseudoprincipal.*

(2) $\Sigma \in \mathcal{M} \Longleftrightarrow \forall \Psi \in \mathfrak{F}_0(\mathfrak{P}_0(X)), \Psi \preceq \Sigma : \exists \Sigma_\Psi \in \mathcal{M} : \Psi \preceq \Sigma_\Psi$ *holds, i.e. a multifilter* Σ *on* X *belongs to* $\mathcal{M}$*, if and only if every refining powerfilter is fine w.r.t.* $\mathcal{M}$*.*

Proof: If $\Sigma \in \mathcal{M}$ holds, the other statement is always fulfilled with $\Sigma_\Psi := \Sigma$. So, let $(X, \mathcal{M})$ be a multifilter-space and let $(\forall \Psi \in \mathfrak{F}_0(\mathfrak{P}_0(X)), \Psi \preceq \Sigma : \exists \Sigma_\Psi \in \mathcal{M} : \Psi \preceq \Sigma_\Psi) \Rightarrow \Sigma \in \mathcal{M}$ hold. Now, let $\Phi \in \mathfrak{F}(\mathfrak{P}_0(X))$ be given with $\forall \Psi \in \mathfrak{F}_0(\Phi) : \Psi \in \mathcal{M}^p$, i.e. $\forall \Psi \in \mathfrak{F}_0(\Phi) : \exists \Sigma_\Psi \in \mathcal{M} : \Psi \preceq \Sigma_\Psi$, by construction of $\mathcal{M}^p$. Of course, these ultrafilters Ψ are just the same, for which $\Psi \preceq \Sigma := [\Phi]_{\widehat{\mathfrak{F}}(X)}$ holds, by proposition 27,(5). So, our assumtion yields $\Sigma \in \mathcal{M}$, implying $\Phi \in \mathcal{M}^p$, by proposition 27(4) and construction of $\mathcal{M}^p$. Thus, $(X, \mathcal{M}^p)$ is pseudoprincipal and so $(X, \mathcal{M})$ is. If otherwise $(X, \mathcal{M})$ is assumed to be pseudoprincipal, and $\forall \Psi \in \mathfrak{F}_0(\mathfrak{P}_0(X)), \Psi \preceq \Sigma : \exists \Sigma_\Psi \in \mathcal{M} : \Psi \preceq \Sigma_\Psi$ holds for some multifilter Σ, then with proposition 27(5) just follows $\Phi := [\Sigma^{\mathfrak{P}_0}]_{\mathfrak{F}(\mathfrak{P}_0(X))} \in \mathcal{M}^p$ and therefore $\Sigma \in (\mathcal{M}^p)^m = \mathcal{M}$ as seen at lemma 59. ■

63 Corollary
Let $(X, \mathcal{M})$ *be a pseudoprincipal multifilter-space.*
Then $\forall n \in I\!N, \Sigma_1, ..., \Sigma_n \in \mathcal{M} : \bigcap_{i=1}^n \Sigma_i \in \mathcal{M}$ *holds, i.e. every pseudoprincipal multifilter-space is limited.*

Proof: Let $\Sigma_1, ...\Sigma_n \in \mathcal{M}$ and $\Phi \in \mathfrak{F}_0(\mathfrak{P}_0(X))$ with $\Phi \preceq \bigcap_{i=1}^n \Sigma_i$ be given and assume $\forall i = 1, ..., n : \Phi \not\preceq \Sigma_i$, i.e. $\forall i = 1, ..., n : \exists \alpha_i \in \Sigma_i : \forall \mathfrak{A} \in \Phi : \mathfrak{A} \not\preceq \alpha_i$. But then $\alpha := \bigcup_{i=1}^n \alpha_i \in \bigcap_{i=1}^n \Sigma_i$ leads to $\exists \mathfrak{A} \in \Phi : \mathfrak{A} \preceq \alpha$, just implying $\mathfrak{A} \subseteq \bigcup_{A \in \alpha} \mathfrak{P}_0(A)$ and consequently $\bigcup_{A \in \alpha} \mathfrak{P}_0(A) = \bigcup_{i=1}^n \left(\bigcup_{A \in \alpha_i} \mathfrak{P}_0(A) \right) \in \Phi$. This would imply $\mathfrak{B} := \bigcup_{A \in \alpha_{i_0}} \mathfrak{P}_0(A) \in \Phi$ for some $i_0 \in \{1, ..., n\}$, by proposition 7. Now, $\mathfrak{B} \preceq \alpha_{i_0}$ in contradiction to our assumption. So, every $\Phi \in \mathfrak{F}_0(\mathfrak{P}_0(X))$ with $\Phi \preceq \bigcap_{i=1}^n \Sigma_i$ must refine some of the $\Sigma_i \in \mathcal{M}$. Now, proposition 62 applies. ■

2.2 Precompactness

64 Definition
Let $(X, \mathcal{M})$ be a powerfilter-space and $\varphi \in \mathfrak{F}(X)$. Then φ is called a **Cauchy-filter** *w.r.t. $\mathcal{M}$ (or $\mathcal{M}$-**Cauchy-filter**), iff*

$$\varphi^{\mathfrak{P}_0} := [\{\mathfrak{P}_0(P) |\ P \in \Phi\}]_{\mathfrak{F}(\mathfrak{P}_0(X))} \in \mathcal{M}$$

holds. The family of all Cauchy filters on X w.r.t. $\mathcal{M}$ we denote by $\gamma_{\mathcal{M}}(X)$.

Obviously, for each powerfilter-structure, all singleton-powerfilters must be Cauchy.

65 Proposition
Let $(X, \mathcal{M}), (Y, \mathcal{N})$ be powerfilter-spaces, φ a Cauchy-filter on X and $f : X \to Y$ a fine map. Then $f(\varphi)$ is a Cauchy-filter on Y.

Proof: For $A \subseteq X$ and $N \in \mathfrak{P}_0(f(A))$ we have always $f(f^{-1}(N) \cap A) = N$, thus $\mathfrak{P}_0(f(A)) \subseteq f(\mathfrak{P}_0(A))$, yielding $f(\varphi)^{\mathfrak{P}_0} \supseteq f(\varphi^{\mathfrak{P}_0}) \in \mathcal{N}$. ■

66 Proposition
If $(X, \mathcal{M})$ is a refinement-closed powerfilter-space and $\varphi \in \mathfrak{F}(X)$, then the following are equivalent:

(1) $\varphi \in \gamma_{\mathcal{M}}(X)$

(2) $\forall \Phi \in \mathfrak{F}(\mathfrak{P}_0(X)) : \Phi \preceq \{\{P\} |\ P \in \varphi\} \Rightarrow \Phi \in \mathcal{M}$.

(3) $\varphi^+ := [\{\varphi \cap \mathfrak{P}_0(P) |\ P \in \varphi\}] \in \mathcal{M}$

(4) $\exists \Sigma \in \mathcal{M} : \forall \alpha \in \Sigma : \varphi \cap \alpha \neq \emptyset$

Proof: (1)⇒(2): Let $\varphi^{\mathfrak{P}_0} \in \mathcal{M}$ and $\Phi \preceq \{\{P\} | P \in \varphi\}$. Then $\Phi \preceq \varphi^{\mathfrak{P}_0}$, because $\{P\} \preceq \mathfrak{P}_0(P)$. Thus, $\Phi \in \mathcal{M}$, because $\mathcal{M}$ is refinement-closed.
(2)⇒(3): We have $\varphi^+ \preceq \{\{P\} | P \in \varphi\}$, because $\varphi \cap \mathfrak{P}_0(P) \preceq \{P\}$.
(3)⇒(4): Take φ^+ as Σ.
(4)⇒(1): Note that $P \in \varphi \cap \alpha$ implies $\mathfrak{P}_0(P) \preceq \alpha$, thus from (4) follows $\varphi^{\mathfrak{P}_0} \preceq \Sigma$ and so $\varphi^{\mathfrak{P}_0} \in \mathcal{M}$, because $(X, \mathcal{M})$ is refinement-closed. ■

If $(X, \mathcal{M})$ is a multifilter-space, we call a filter φ on X a *Cauchy-filter*, iff it is Cauchy in the corresponding powerfilter-space. Because of 66(4), this is equivalent to $\exists \Sigma \in \mathcal{M} : \forall \alpha \in \Sigma : \varphi \cap \alpha \neq \emptyset$. The family of all Cauchy-filters is denoted by $\gamma_{\mathcal{M}}$ in this case, too.

67 Definition
A powerfilter-space $(X, \mathcal{M})$ is called **precompact**, *iff $\mathfrak{F}_0(X) \subseteq \gamma_{\mathcal{M}}(X)$, i.e. every ultrafilter on X is Cauchy w.r.t. $\mathcal{M}$. A subset A of X is said to be precompact (in $(X, \mathcal{M})$), iff it is precompact as a subspace. i.e. $(A, \mathcal{M}_{|A})$ is precompact in the above*

sense. Furthermore, a filter $\varphi \in \mathfrak{F}(X)$ *is called* precompact, *iff* $\mathfrak{F}_0(\varphi) \subseteq \gamma_{\mathcal{M}}(X)$.
A subset or a filter in a multifilter-space $(X, \mathcal{M})$ *is said to be precompact, iff it is precompact in the corresponding refinement-closed powerfilter-space* $(X, \mathcal{M}^p)$.
By $\mathcal{PC}(X)$ *we denote the family of all nonempty precompact subsets of a powerfilter-space or a multifilter-space.*

Because of proposition 46, it's clear, that a subset A is precompact in $(X, \mathcal{M})$, iff every ultrafilter on X, which contains A, is Cauchy in $(X, \mathcal{M})$.

68 Proposition
Let $(X, \mathcal{M}), (Y, \mathcal{N})$ *be powerfilter-spaces and* $f : X \to Y$ *a fine map. If* $(X, \mathcal{M})$ *is precompact, then its image* $f(X)$ *is precompact in* $(Y, \mathcal{N})$.

Proof: Let $\psi \in \mathfrak{F}_0(f(X))$, then there exists an ultrafilter φ on X with $f(\varphi) = \psi$, by corollary 11. Now, φ is Cauchy by the assumption of precompactness for X, so $\psi = f(\varphi)$ is Cauchy by proposition 65. ■

69 Corollary
Let $(X, \mathcal{M}), (Y, \mathcal{N})$ *be multifilter-spaces and* $f : X \to Y$ *a fine map. If* $(X, \mathcal{M})$ *is precompact, then its image* $f(X)$ *is precompact in* $(Y, \mathcal{N})$.

Proof: Follows from the preceding proposition and lemma 59. ■

70 Theorem
(Tychonoff)
Let $(X_i, \mathcal{M}_i)_{i \in I}$ *be a family of powerfilter-spaces. Then the product* $\prod_{i \in I}(X_i, \mathcal{M}_i)$ *is precompact if and only if all* $(X_i, \mathcal{M}_i)$ *are precompact.*

Proof: If the product is precompact, then the precompactness of all $(X_i, \mathcal{M}_i)$ follows from proposition 68, because all canonical projections are fine and surjective. Now, let all $(X_i, \mathcal{M}_i)$ be precompact and $\Phi \in \mathfrak{F}_0(\mathfrak{P}_0(\prod_{i \in I} X_i))$. Then for every canonical projection $p_i, i \in I$, holds $p_i(\Phi) \in \mathfrak{F}_0(X_i)$, thus Φ belongs to the initial powerfilter-structure w.r.t. these projections, by proposition 43, which is just the product structure, by [43], Th. 1.2.1.3. ■

71 Corollary
Let $(X_1, \mathcal{M}_i)_{i \in I}$ *be a family of multifilter-spaces. The product* $\prod_{i \in I}(X_i, \mathcal{M}_i)$ *is precompact if and only if all* $(X_i, \mathcal{M}_i)$ *are precompact.*

Proof: Follows from the preceding theorem, lemma 59, proposition 50 and [43], Th. 2.2.12, 2.2.13(2). ■

A slight modification of an usual argument for compactness-like properties now leads to a slight modification of an usual description of precompactness in the covering sense.

72 Lemma
Let $(X, \mathcal{M})$ be a refinement-closed powerfilter-space and $P \subseteq X$. Then P is precompact, iff

$$\exists \Phi_1, ..., \Phi_n \in \mathcal{M} : \forall \alpha \in \bigcap_{i=1}^{n} \Phi_i : \exists A_1, ..., A_m \in \alpha : P \subseteq \bigcup_{j=1}^{m} A_j \ . \tag{1}$$

Proof: Let P be precompact and assume the contrary of (1), i.e. $\forall \Phi_1, ..., \Phi_n \in \mathcal{M} : \exists \alpha_i \in \Phi_i : \forall A_1, ..., A_m \in \bigcup_{i=1}^{n} \alpha_i : P \setminus \bigcup_{j=1}^{m} A_j \neq \emptyset$. Then these $P \setminus \bigcup_{j=1}^{m} A_j$ form a filter base, contained in an ultrafilter φ on P, which must be Cauchy, i.e. $\varphi^{\mathfrak{P}_0} \in \mathcal{M}$. Then by assumption we find $M \in \varphi$, s.t. $\forall A \in \mathfrak{P}_0(M) : P \setminus A \neq \emptyset$, especially for $A = M \in \varphi$, but then we have $P \setminus M \in \varphi$ because of our choice for φ - a contradiction.
Otherwise, let (1) hold, the $\Phi_1, ..., \Phi_n$ be given and let φ be an ultrafilter on P. Now, assume $\forall i = 1, ..., n : \exists \alpha_i \in \Phi_i : \alpha_i \cap \varphi = \emptyset$. Then $\varphi \cap \bigcup_{i=1}^{n} \alpha_i = \emptyset$ follows, but otherwise there are $A_1, ..., A_m \in \bigcup_{i=1}^{n} \alpha_i$ s.t. $P \subseteq \bigcup_{j=1}^{m} A_j$, implying that φ contains one of these A_j by proposition 7 - a contradiction. Thus, there must be one of the Φ_i, say Φ_1, with $\forall \alpha \in \Phi_1 : \alpha \cap \varphi \neq \emptyset$. This implies $\varphi^{\mathfrak{P}_0} \preceq \Phi_1$ and thus $\varphi^{\mathfrak{P}_0} \in \mathcal{M}$ by refinement-closedness. ■

73 Corollary
Let $(X, \mathcal{M})$ be a multifilter-space and $P \subseteq X$. Then P is precompact, iff

$$\exists \Sigma_1, ..., \Sigma_n \in \mathcal{M} : \forall \alpha \in \bigcap_{i=1}^{n} \Sigma_i : \exists A_1, ..., A_m \in \alpha : P \subseteq \bigcup_{j=1}^{m} A_j \ . \tag{2}$$

Proof: Remember, that $\mathcal{M}^p$ is build from refining powerfilters of the multifilters $\Sigma \in \mathcal{M}$ and conversely, every powerfilter from $\mathcal{M}^p$ is refined by some multifilter from $\mathcal{M}$. ■

74 Corollary
Let $(X, \mathcal{M})$ be a limited multifilter-space and $P \subseteq X$. Then P is precompact, iff $\exists \Sigma \in \mathcal{M} : \forall \alpha \in \Sigma : \exists A_1, ..., A_m \in \alpha : P \subseteq \bigcup_{j=1}^{m} A_j$.

Proof: Follows directly from lemma 72 and definition 51. ■

From this, it's easily seen, that our notion of precompactness on **UMer** coincides with the usual one for uniform covering spaces, meaning that a uniform principal multifilterspace is precompact if and only if its corresponding uniform covering space is precompact in the classical sense.

75 Proposition
Let $(X,\mathcal{M}) \in |\,\mathbf{PFS}\,|$, $A \subseteq X$ and $\varphi,\psi \in \mathfrak{F}(X)$. Then hold

(1) If $\varphi \in \gamma_{\mathcal{M}}(X)$ and $\psi \supseteq \varphi$, then $\psi \in \gamma_{\mathcal{M}}(X)$.

(2) If $\psi \supseteq \varphi$ and φ is precompact, then ψ is precompact, too.

(3) A is precompact, iff the principal filter $[A]$ is precompact.

(4) If A is precompact, then each subset of A is precompact, too.

(5) If $(Y,\mathcal{N})$ is a powerfilterspace, too, φ is a precompact filter on X and $f \in Y^X$ is fine w.r.t. $\mathcal{M},\mathcal{N}$, then $f(\varphi)$ is a precompact filter on Y.

Proof: (1) is a consequence of the obvious fact, that $\psi \supseteq \varphi$ implies $\psi^{\mathfrak{P}_0} \supseteq \varphi^{\mathfrak{P}_0}$; to verify (2), remember $\mathfrak{F}_0(\psi) \subseteq \mathfrak{F}_0(\varphi) \subseteq \gamma_{\mathcal{M}}(X)$; (3) comes from $\mathfrak{F}_0(A) = \mathfrak{F}_0([A])$ and (4) follows from (2) and (3). (5) follows directly from corollary 11 and the definitions of precompactness and fine maps. ■

The precompact refinement-closed powerfilter-spaces and fine maps between them form obviously a (full and isomorphism-closed) subcategory of $\mathbf{PFS}^{\preceq}$, which we denote by $\mathbf{pcPFS}^{\preceq}$.

76 Lemma
$\mathbf{pcPFS}^{\preceq}$ *is a bireflective subcategory of* $\mathbf{PFS}^{\preceq}$.

Proof: For $(X,\mathcal{M}) \in |\,\mathbf{PFS}^{\preceq}\,|$ define $\mathcal{M}^{pc} := \mathcal{M} \cup \{\Phi \in \mathfrak{F}(\mathfrak{P}_0(X))|\ \exists \varphi \in \mathfrak{F}_0(X) : \Phi \preceq \{\{P\}|\ P \in \varphi\}\}$. Then clearly $\mathcal{M} \subseteq \mathcal{M}^{pc}$, implying $\mathbb{1}_X : (X,\mathcal{M}) \to (X,\mathcal{M}^{pc})$ to be fine, and for an arbitrary precompact refinement-closed powerfilter-space $(Y,\mathcal{N})$ and a map $f : X \to Y$ with $f(\mathcal{M}) \subseteq \mathcal{N}$ we have $f(\mathcal{M}^{pc}) = f(\mathcal{M}) \cup f(\{\Phi \in \mathfrak{F}(\mathfrak{P}_0(X))|\ \exists \varphi \in \mathfrak{F}_0(X) : \Phi \preceq \{\{P\}|\ P \in \varphi\}\})$. Here we see $f(\{\Phi \in \mathfrak{F}(\mathfrak{P}_0(X))|\ \exists \varphi \in \mathfrak{F}_0(X) : \Phi \preceq \{\{P\}|\ P \in \varphi\}\}) = \{f(\Phi)|\ \Phi \in \mathfrak{F}(\mathfrak{P}_0(X)) : \exists \varphi \in \mathfrak{F}_0(X) : \Phi \preceq \{\{P\}|\ P \in \varphi\}\} \subseteq \{\Psi \in \mathfrak{F}(\mathfrak{P}_0(Y))|\ \exists \varphi \in \mathfrak{F}_0(X) : \Psi \preceq \{\{f(P)\}|\ P \in \varphi\}\} \subseteq \{\Psi \in \mathfrak{F}(\mathfrak{P}_0(Y))|\ \exists \psi \in \mathfrak{F}_0(Y) : \Psi \preceq \{\{P\}|\ P \in \psi\}\} \subseteq \mathcal{N}$, because of proposition 15, corollary 11 and the precompactness of the refinement-closed $(Y,\mathcal{N})$, respectively. ■

77 Proposition
Let $(X,\mathcal{M})$ be a pseudoprincipal and refinement-closed powerfilter-space and let $\varphi \in \mathfrak{F}(X)$ be precompact. Then the powerfilter $[\varphi^{\{\}}]$ belongs to $\mathcal{M}$.

Proof: Let $\Phi \in \mathfrak{F}_0([\varphi^{\{\}}])$. Then $\Phi^{\cup}$ with the map $^{\cup} : \{\{x\}|x \in X\} \to X : \{x\}^{\cup} \to x$ is an ultrafilter on X (by corollary 11), which obviously refines φ. Now, because of the precompactness of φ, this $\Phi^{\cup}$ must be Cauchy, i.e. $\exists \Sigma \in \mathcal{M} : \forall \alpha \in \Sigma : \exists B \in \Phi^{\cup} : B \in \alpha$, which implies $B^{\{\}} \preceq \alpha$. Thus $(\Phi^{\cup})^{\{\}} \preceq \Sigma$, implying $(\Phi^{\cup})^{\{\}} \in \mathcal{M}$ by refinement-closedness. But $(\Phi^{\cup})^{\{\}} = \Phi$, by proposition 23(3). This holds for every

ultrafilter Φ, which is finer than $[\varphi^{\{\}}]$, thus $[\varphi^{\{\}}] \in \mathcal{M}$, because $(X, \mathcal{M})$ is pseudo-principal. ■

78 Definition
Let $(X, \mathcal{M})$ be a multifilter- or powerfilter-space. It is called **locally precompact**, *iff all members of $\mathcal{M}$ contain a partial cover, whose union is precompact.*

Obviously, a multifilter-space $(X, \mathcal{M})$, whose structure contains the multifilter $[X^{\{\}}]$, is locally precompact, iff it is precompact.

2.3 Convergence for Multifilter-Spaces

79 Definition
Let $(X, \mathcal{M})$ be a multifilter-space. Then a generalized convergence structure $q_{\gamma_\mathcal{M}}$ is defined on X by

$$q_{\gamma_\mathcal{M}} := \{(\varphi, x) \in \mathfrak{F}(X) \times X |\ \varphi \cap \overset{\bullet}{x} \in \gamma_\mathcal{M}(X)\} \ .$$

From the definition follows at once, that every filter on X, which converges w.r.t. $q_{\gamma_\mathcal{M}}$, must be $\mathcal{M}$-Cauchy. Furthermore, it is obvious, that this convergence on **PrULimMFS** coincides with the usual convergence in uniform spaces, i.e. a filter on a set X converges w.r.t. to a principal uniform multifilter-structure, iff it converges w.r.t. the corresponding uniform covering structure (in the sense of Tukey).

80 Proposition
If $(X, \mathcal{M})$ is a multifilter-space, then $(X, q_{\gamma_\mathcal{M}})$ is a symmetric Kent-convergence space.

Proof: It is a Kent-convergence space, because trivially $\varphi \cap \overset{\bullet}{x} = \varphi \cap \overset{\bullet}{x} \cap \overset{\bullet}{x}$ holds. To verify symmetry, let $(\varphi, x) \in q_{\gamma_\mathcal{M}}, y \in X$ with $\overset{\bullet}{y} \supseteq \varphi$ be given. But then follows $\varphi = \varphi \cap \overset{\bullet}{y}$, thus $\varphi \cap \overset{\bullet}{y} \supseteq \varphi \cap \overset{\bullet}{x} \in \gamma_\mathcal{M}$ and consequently $(\varphi, y) \in q_{\gamma_\mathcal{M}}$. ■

81 Proposition
Let $(X, \mathcal{M}), (Y, \mathcal{N})$ be multifilter-spaces and $f : X \to Y$ a fine map w.r.t. $\mathcal{M}, \mathcal{N}$. Then f is continuous w.r.t. $q_{\gamma_\mathcal{M}}, q_{\gamma_\mathcal{N}}$.

Proof: If $(\varphi, x) \in q_{\gamma_\mathcal{M}}$, then $\varphi \cap \overset{\bullet}{x}$ is $\mathcal{M}$-Cauchy by definition, thus $f(\varphi \cap \overset{\bullet}{x}) = f(\varphi) \cap \overset{\bullet}{f(x)}$ is $\mathcal{N}$-Cauchy by proposition 65 (remember lemma 59, too). This yields $(f(\varphi), f(x)) \in q_{\gamma_\mathcal{N}}$ by definition, again. ■

82 Lemma
Let $(X, \mathcal{M})$ be a multifilter-space. Then are equivalent

(1) $(X, q_{\gamma_{\mathcal{M}}})$ *is* T_0,

(2) $(X, q_{\gamma_{\mathcal{M}}})$ *is* T_1 .

If $(X, \mathcal{M})$ *is a weakly uniform limited multifilter-space, then both are equivalent to*

(3) $(X, q_{\gamma_{\mathcal{M}}})$ *is* T_2 .

Proof: The equivalence of (1) and (2) follows directly from propositions 33, 80. (3)⇒(2) is trivial, so let (2) be valid and $\varphi \in \mathfrak{F}(X), x, y \in X$ be given with $(\varphi, x) \in q_{\gamma_{\mathcal{M}}}$ and $(\varphi, y) \in q_{\gamma_{\mathcal{M}}}$. Then there exist $\Sigma_1, \Sigma_2 \in \mathcal{M}$ with $\varphi \cap \overset{\bullet}{x} \in \gamma_{\Sigma_1}, \varphi \cap \overset{\bullet}{y} \in \gamma_{\Sigma_2}$ and we have $\Sigma := (\Sigma_1 \cap \Sigma_2)^{\diamond} \in \mathcal{M}$. Now, for every member σ of Σ, there are $\sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2$ with $(\sigma_1 \cup \sigma_2)^{\diamond} \preceq \sigma$ and from the convergence of φ follows the existence of $S_1 \in \sigma_1, S_2 \in \sigma_2$ with $x \in S_1 \in \varphi, y \in S_2 \in \varphi$. Because S_1, S_2 both belong to the filter φ, there is $z \in S_1 \cap S_2$ and consequently $x, y \in S_1 \cup S_2 \in \diamond(z, \sigma_1 \cup \sigma_2) \subseteq (\sigma_1 \cup \sigma_2)^{\diamond} \preceq \sigma$. So, there exists $S \in \sigma$ with $S \in \overset{\bullet}{x} \cap \overset{\bullet}{y}$. Thus $(\overset{\bullet}{x}, y) \in q_{\gamma_{\mathcal{M}}}$ follows, implying $x = y$ by (2). ■

2.4 Completeness and Compactness

83 Definition
A multifilter-space $(X, \mathcal{M})$ *is said to be* **complete**, *iff all Cauchy-filters w.r.t.* $\mathcal{M}$ *converge w.r.t.* $q_{\gamma_{\mathcal{M}}}$. *A subset of* X *is called complete (w.r.t.* $\mathcal{M}$*), iff it is complete as a subspace.*

84 Proposition
Let $(X, \mathcal{M})$ *be a weakly uniform limited multifilter-space,* φ *a Cauchy-filter on* X *with an adherence point* $x \in X$, *w.r.t.* $q_{\gamma_{\mathcal{M}}}$. *Then* φ *converges to* x *w.r.t.* $q_{\gamma_{\mathcal{M}}}$.

Proof: Let $\varphi \in \gamma_{\mathcal{M}}, x \in X$ be given with $\varphi_0 \in \mathfrak{F}_0(\varphi), (\varphi_0, x) \in q_{\gamma_{\mathcal{M}}}$. Then there are $\Sigma_1, \Sigma_2 \in \mathcal{M}$ with $\varphi \in \gamma_{\Sigma_1}$, $\varphi_0 \cap \overset{\bullet}{x} \in \gamma_{\Sigma_2}$. Now, take $\Sigma := (\Sigma_1 \cap \Sigma_2)^{\diamond} \in \mathcal{M}$. For every $\sigma \in \Sigma$ exist $\sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2$ s.t. $(\sigma_1 \cup \sigma_2)^{\diamond} \preceq \sigma$. There is $S_2 \in \sigma_2$, which contains x and belongs to φ_0, and $S_1 \in \sigma_1 \cap \varphi$. Because S_1, S_2 both belong to φ_0, there exists $z \in S_1 \cap S_2$, implying $x \in S_1 \cup S_2 \in \diamond(z, \sigma_1 \cup \sigma_2) \subseteq (\sigma_1 \cup \sigma_2)^{\diamond} \preceq \sigma$, thus exists $S \in \sigma$ with $x \in S \in \varphi$, because $S_1 \cup S_2 \in \varphi$. ■

We will call a multifilter-space $(X, \mathcal{M})$ *compact*, iff $(X, q_{\gamma_{\mathcal{M}}})$ is compact.

85 Lemma
(1) Every precompact and complete multifilter-space is compact.

(2) (a) Every compact multifilter-space is precompact.

(b) Every compact weakly uniform limited multifilter-space is complete.

Proof: (1): Follows just from combining the definitions of precompactness and completeness. (2)(a): Follows, because on a compact space every ultrafilter converges, and thus must be Cauchy, as mentioned above. (2)(b): Follows from proposition 84, because every Cauchy-filter has an adherence point, if its refining ultrafilters converge - and here they do, by compactness. ■

3 Function Spaces

3.1 Topological Base Spaces

A very interesting and fairly wide class of function space structures, defined for Y^X or $C(X,Y)$ with a set X and a topological space (Y,σ), are the so called set–open topologies, examined in [39].

86 Definition
(see [39], (2.26))
Let X and Y be sets and $A \subseteq X$, $B \subseteq Y$; then let be $(A,B) := \{f \in Y^X |\ f(A) \subseteq B\}$. Now let X be a set, (Y,σ) a topological space and $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$. Then the topology $\tau_{\mathfrak{A}}$ on Y^X (resp. $C(X,Y)$), which is defined by the open subbase $\{(A,W) |\ A \in \mathfrak{A}, W \in \sigma\}$ is called the **set–open topology, generated by $\mathfrak{A}$**, *or shortly the* **$\mathfrak{A}$–open topology**.

By $\mathfrak{F}(X)_{\mathfrak{A}}$ we denote the set of all filters on X, which have a base, consisting of elements of $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$.

87 Proposition
Let X be a set, (Y,σ) a topological space and $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$, $\mathcal{F} \in \mathfrak{F}(Y^X)$, $f \in Y^X$. Then holds

$$(\mathcal{F},f) \in q_{\tau_{\mathfrak{A}}} \Longleftrightarrow \forall \varphi \in \mathfrak{F}(X)_{\mathfrak{A}} : (\mathcal{F}(\varphi), f(\varphi)) \in \tilde{q_\sigma}\ .$$

Proof: Let $(\mathcal{F},f) \in q_{\tau_{\mathfrak{A}}}$ and $\varphi \in \mathfrak{F}(X)_{\mathfrak{A}}$. For any $W \in \sigma \cap f(\varphi)$ there is an $A \in \mathfrak{A}$, such that $f(A) \subseteq W$, because of $\varphi \in \mathfrak{F}(X)_{\mathfrak{A}}$. This means $f \in (A,W) \in \tau_{\mathfrak{A}}$, implying $(A,W) \in \mathcal{F}$ by $\mathcal{F} \xrightarrow{\tau_{\mathfrak{A}}} f$. So, we have $W \supseteq \omega(A,(A,W)) \in \mathcal{F}(\varphi)$.
If $\forall \varphi \in \mathfrak{F}(X)_{\mathfrak{A}} : (\mathcal{F}(\varphi), f(\varphi)) \in \tilde{q_\sigma}$ holds, we may chose the principal filters $[A]$ with $A \in \mathfrak{A}$ for φ to get $\mathcal{F}(A) \subseteq W$ for all $W \in \sigma \cap f(A)$, implying $(A,W) \in \mathcal{F}$ for any $A \in \mathfrak{A}, W \in \sigma$. ∎

Now, we extend the class of the set–open topologies on $C(X,Y)$ to a greater class of convergence structures.

88 Definition
Let (X,τ), (Y,σ) be topological spaces and $\tilde{\mathfrak{A}} \subseteq \mathfrak{F}(X)$. Then we call

$$q_{\tilde{\mathfrak{A}}} := \left\{(\mathcal{F},f) \in \mathfrak{F}(C(X,Y)) \times C(X,Y) |\ \forall \varphi \in \tilde{\mathfrak{A}} : (\mathcal{F}(\varphi), f(\varphi)) \in \tilde{q_\sigma}\right\}$$

the structure of **$\tilde{\mathfrak{A}}$–continuous convergence** *for $C(X,Y)$.*

Obviously, every convergence, generated from a set-open topology $\tau_{\mathfrak{A}}$ coincides with the structure of $\mathfrak{F}(X)_{\mathfrak{A}}$-continuous convergence on $C(X,Y)$, just by proposition 87.

89 Definition
Let (X,q) be a convergence space and $\varphi \in \mathfrak{F}(X)$. Then φ is said to be **compactoid**[5] *w.r.t. q, iff*

$$\forall \varphi_0 \in \mathfrak{F}_0(\varphi), P \in \varphi : P \cap q(\varphi_0) \neq \emptyset \ ,$$

i.e. for every refining ultrafilter of φ, every member of φ contains an element, to which the ultrafilter converges.
The set of all compactoid filters on X w.r.t. q is denoted by $\mathfrak{C}(X)_q$, or, if no misunderstanding should be to aware, simply by $\mathfrak{C}(X)$.

Obviously, all compactly generated filters are compactoid, and - for pretopological spaces - all neighbourhood-filters are compactoid, too.

90 Lemma
Let (X,τ) be a topological space and φ a filter on X. Then φ is compactoid w.r.t. q_τ, iff for every family $(O_i)_{i\in I}$ of τ-open subsets O_i of X

$$\bigcup_{i\in I} O_i \in \varphi \iff \exists n \in I\!N, i_1, ..., i_n \in I : \bigcup_{k=1}^{n} O_{i_k} \in \varphi$$

holds.

Proof: Let $\varphi \in \mathfrak{F}(X)$ be compactoid and an arbitrary family $(O_i)_{i\in I}$ of open sets with $\bigcup_{i\in I} O_i \in \varphi$ be given. Assume $\forall J \subseteq I, card(J) \in I\!N : \bigcup_{j\in J} O_j \notin \varphi$, just meaning $\forall P \in \varphi : P \cap (\bigcup_{j\in J} O_j)^c \neq \emptyset$, consequently the filter-base $\mathfrak{B} := \{X \setminus \bigcup_{j\in J} O_j |\ J \subseteq I, card(J) \in I\!N\}$ is compatible with φ and so, there exists an ultrafilter φ_0, which contains both, φ and $\mathfrak{B}$. Then φ_0 converges especially on $\bigcup_{i\in I} O_i$ to a point x_0, because of the compactoidness of φ. Now, x_0 belongs to at least one of the open sets, say $x \in O_{i_x}$, which therefore must be contained in φ_0 - in contradiction to the fact, that φ_0 should contain $X \setminus O_{i_x}$ by construction.
Otherwise, let $\varphi \in \mathfrak{F}(X)$ be given with the property, that it contains a finite union of elements of every collection of open sets, whose union is contained in φ. Assume, there would exist a refining ultrafilter φ_0 on φ, which doesn't converge on some element P of φ. Then every point $p \in P$ has an open neighbourhood O_p, which is not contained in φ_0. But $\bigcup_{p\in P} O_p$ is an element of φ, because P is, and so there must exist a finite subset $p_1, ..., p_n$ of P s.t. $\bigcup_{k=1}^{n} O_{p_k} \in \varphi \subseteq \varphi_0$. But then, by proposition 7, φ_0 must contain one of these O_{p_k} - a contradiction. ■

[5] In [13], Dolecki deals with filters, which he called *compactoid in a set A*, so he gets a relative notion, depending on *one special* reference-set A, whereas our compactoidness always refers exactly to *all members* of the filter in question. So we get a somewhat stronger condition and a more absolute notion. The difference between these two notions is verbally expressed just as the absence of a reference-set in our formulation.

91 Proposition
If $(X,\tau),(Y,\sigma)$ are topological spaces, $f \in C(X,Y)$ and $\varphi \in \mathfrak{C}(X)$, then $f(\varphi) \in \mathfrak{C}(Y)$.

Proof: If $\psi_0 \in \mathfrak{F}_0(f(\varphi))$, then there exists by corollary 11 an ultrafilter $\varphi_0 \in \mathfrak{F}_0(\varphi)$ with $f(\varphi_0) = \psi_0$. Because of the compactoidness of φ, this ultrafilter converges on every member of φ, thus by continuity of f, the image $f(\varphi_0) = \psi_0$ converges on every image $f(A)$ with $A \in \varphi$. But these images form a base for $f(\varphi)$. ■

To use the word "continuous"in definition 88, may be justified by the following.

92 Lemma
Let $(X,\tau),(Y,\sigma)$ be topological spaces and $\tilde{\mathfrak{A}} \subseteq \mathfrak{F}(X)$. Then holds:

(1) If all members of $\tilde{\mathfrak{A}}$ are compactoid, then $q_{\tilde{\mathfrak{A}}}$ is splitting, i.e. $q_c \subseteq q_{\tilde{\mathfrak{A}}}$.

(2) If $\tilde{\mathfrak{A}} \supseteq \{\underline{U}(x) |\ x \in X\}$, then $q_{\tilde{\mathfrak{A}}}$ is conjoining, i.e. $q_{\tilde{\mathfrak{A}}} \subseteq q_c$.

(3) If $\{\underline{U}(x) |\ x \in X\} \subseteq \tilde{\mathfrak{A}} \subseteq \mathfrak{C}(X)$, then $q_{\tilde{\mathfrak{A}}} = q_c$.

Proof: (1): Let $(\mathcal{F},f) \in q_c$, $\varphi \in \tilde{\mathfrak{A}}$ and $V \in f(\varphi) \cap \sigma$. By lemma 10, for every $\psi_0 \in \mathfrak{F}_0(\mathcal{F}(\varphi))$, there are $\mathcal{F}_0 \in \mathfrak{F}_0(\mathcal{F}), \varphi_0 \in \mathfrak{F}_0(\varphi)$ such that $\mathcal{F}_0(\varphi_0) \subseteq \psi_0$. Now, $f^{-1}(V) \in \varphi$ and φ is compactoid, thus $\exists x_0 \in f^{-1}(V) : (\varphi_0, x_0) \in q_\tau$. This implies $(\mathcal{F}_0(\varphi_0), f(x_0)) \in q_\sigma$, because $\mathcal{F}$ converges continuously to f, and so $\mathcal{F}_0$ does. The given V is an open neighbourhood of $f(x_0)$, thus $V \in \mathcal{F}_0(\varphi_0) \subseteq \psi_0$. So, every refining ultrafilter of $\mathcal{F}(\varphi)$ contains V and therefore $V \in \mathcal{F}(\varphi)$. This holds for all $V \in f(\varphi) \cap \sigma$, implying $\mathcal{F}(\varphi) \supseteq f(\varphi) \cap \sigma$. This is valid for all $\varphi \in \tilde{\mathfrak{A}}$, yielding $(\mathcal{F},f) \in q_{\tilde{\mathfrak{A}}}$.
(2): Given $(\mathcal{F},f) \in q_{\tilde{\mathfrak{A}}}$ and any $(\varphi,x) \in q_\tau$, we have $\varphi \supseteq \underline{U}(x)$, implying $\mathcal{F}(\varphi) \supseteq \mathcal{F}(\underline{U}(x)) \supseteq f(\underline{U}(x)) \cap \sigma$ by $\tilde{\mathfrak{A}}$-continuous convergence of $\mathcal{F}$ to f. By the continuity of f we get $f(\underline{U}(x)) \supseteq \overset{\bullet}{f(x)} \cap \sigma$, thus $\mathcal{F}(\varphi) \supseteq \overset{\bullet}{f(x)} \cap \sigma \cap \sigma = \overset{\bullet}{f(x)} \cap \sigma$, yielding $(\mathcal{F}(\varphi), f(x)) \in q_\sigma$ and now, because this holds for every $(\varphi,x) \in q_\tau$, we have $(\mathcal{F},f) \in q_c$.
(3): follows immediately from (1) and (2), because the neighbourhood-filters are all compactoid. ■

3.2 Function Spaces in PFS and MFS

93 Proposition
Let $(X,\mathcal{M}) \in |\,\mathbf{PFS}\,|$ and $(Y,\mathcal{M}) \in |\,\mathbf{psPFS}\,|$. Then $(\mathbf{Y^X}, \mathcal{M}_{\mathbf{X,Y}})$ is pseudoprincipal, too.

Proof: Suppose $\Omega \in \mathfrak{F}(\mathfrak{P}_0(\mathbf{Y^X})), \Omega \notin \mathcal{M}_{\mathbf{Y,X}}$. Then $\exists \Sigma \in \mathcal{M} : \Omega(\Sigma) \notin \mathcal{N}$, implying $\exists \Xi \in \mathfrak{F}_0(\Omega(\Sigma)) : \Xi \notin \mathcal{N}$, because $(Y,\mathcal{N})$ is pseudoprincipal. Now, by lemma

10 there are $\Omega' \in \mathfrak{F}_0(\Omega), \Sigma' \in \mathfrak{F}_0(\Sigma)$ such that $\Omega'(\Sigma') \subseteq \Xi$, implying $\Omega'(\Sigma') \notin \mathcal{N}$, because $\Xi \notin \mathcal{N}$. But $\Sigma' \supseteq \Sigma \in \mathcal{M}$, thus $\Omega' \notin \mathcal{M}_{\mathbf{X},\mathbf{Y}}$. ■

94 Proposition
Let $(X,\mathcal{M}) \in |\mathbf{PFS}|$ and $(Y,\mathcal{M}) \in |\mathbf{PFS}^{\preceq}|$. Then $(\mathbf{Y}^{\mathbf{X}}, \mathcal{M}_{\mathbf{X},\mathbf{Y}})$ is refinement-closed, too.

Proof: Let $\Gamma \in \mathcal{M}_{\mathbf{X},\mathbf{Y}}$, i.e. $\forall \Phi \in \mathcal{M} : \Gamma(\Phi) \in \mathcal{N}$. Now, for $\Gamma' \preceq \Gamma$, we get $\forall \Phi \in \mathcal{M} : \Gamma'(\Phi) \preceq \Gamma(\Phi)$ and consequently $\forall \Phi \in \mathcal{M} : \Gamma'(\Phi) \in \mathcal{N}$, because $(Y,\mathcal{N})$ is refinement-closed. Thus $\Gamma' \in \mathcal{M}_{\mathbf{X},\mathbf{Y}}$. ■

95 Proposition
Let $(X,\mathcal{M}),(Y,\mathcal{N})$ be multifilter-spaces and $\mathcal{F} \in \mathfrak{F}(\mathbf{Y}^{\mathbf{X}})$. Then

$$\mathcal{F} \in \gamma_{\mathcal{M}_{\mathbf{X},\mathbf{Y}}}(\mathbf{Y}^{\mathbf{X}})$$
$$\Longleftrightarrow$$
$$\forall \Sigma \in \mathcal{M} : \exists \Xi \in \mathcal{N} : \forall \xi \in \Xi : \exists \sigma \in \Sigma, F \in \mathcal{F} : F(\sigma) := \{\omega(S \times F) | S \in \sigma\} \preceq \xi$$

holds.

Proof: $\mathcal{F} \in \gamma(\mathcal{M}_{\mathbf{X},\mathbf{Y}}) \Leftrightarrow \widehat{\mathcal{F}} \in \mathcal{M}_{\mathbf{X},\mathbf{Y}} \Leftrightarrow \forall \Sigma \in \mathcal{M} : \widehat{\mathcal{F}}(\Sigma) \in \mathcal{N} \Leftrightarrow \forall \Sigma \in \mathcal{M} : [\{\{F\}(\sigma) | ; \{F\} \in \widehat{\mathcal{F}}, \sigma \in \Sigma\}] = [\{F(S) |\ F \in \mathcal{F}, S \in \sigma\}] =: \Xi \in \mathcal{N}$. ■

96 Definition
Let $(X,\mathcal{M}),(Y,\mathcal{N})$ be multifilter-spaces. A subset $\mathcal{H} \subseteq \mathbf{Y}^{\mathbf{X}}$ is called **equiuniformly fine**, *iff*

$$[\mathcal{H}^{\{\}}] \in \mathcal{M}_{\mathbf{X},\mathbf{Y}}$$

holds, i.e. $[\mathcal{H}^{\{\}}](\mathcal{M}) \subseteq \mathcal{N}$, where $[\mathcal{H}^{\{\}}](\mathcal{M}) := \{[\mathcal{H}^{\{\}}](\Sigma) |\ \Sigma \in \mathcal{M}\}$, $[\mathcal{H}^{\{\}}](\Sigma) := [\{[\mathcal{H}^{\{\}}](\alpha) |\ \alpha \in \Sigma\}]$ and $[\mathcal{H}^{\{\}}](\alpha) := \{f(A) |\ f \in \mathcal{H}, A \in \alpha\}$.
Furthermore, a filter $\mathcal{F} \in \mathfrak{F}(\mathbf{Y}^{\mathbf{X}})$ is called **equiuniformly fine**, *iff $\mathcal{F}^{\{\}} := [\{\mathcal{H}^{\{\}} |\ \mathcal{H} \in \mathcal{F}\}] \in \mathcal{M}_{\mathbf{X},\mathbf{Y}}$.*

97 Lemma
Let $(X,\mathcal{M}) \in |\mathbf{PFS}|$ and $(Y,\mathcal{N})$ be a pseudoprincipal and refinement-closed power-filter-space. Then for each precompact filter $\mathcal{F} \in \mathfrak{F}(\mathbf{Y}^{\mathbf{X}})$ hold:

(1) $\mathcal{F}$ is equiuniformly fine and

(2) For every precompact filter φ on X is $\mathcal{F}(\varphi) := [\{\omega(P \times \mathcal{H}) |\ P \in \varphi, \mathcal{H} \in \mathcal{F}\}]$ a precompact filter on $(Y,\mathcal{N})$.

Proof: (1): By the propositions 93 and 94 we know, that $(\mathbf{Y}^{\mathbf{X}}, \mathcal{M}_{\mathbf{X},\mathbf{Y}})$ is refinement-closed and pseudoprincipal. So, proposition 77 is applicable, yielding that $\mathcal{F}$ is equiuniformly fine.
(2): Let $\varphi \in \mathfrak{F}(X)$ be precompact. For all $\psi \in \mathfrak{F}_0(\mathcal{F}(\varphi))$ we find: By lemma 10 exist $\mathcal{F}_0 \in \mathfrak{F}_0(\mathcal{F}), \varphi_0 \in \mathfrak{F}_0(\varphi)$ such that $\mathcal{F}_0(\varphi_0) \subseteq \psi$. Now, $\mathcal{F}_0$ and φ_0 must be Cauchy w.r.t. $\mathcal{M}_{\mathbf{X},\mathbf{Y}}, \mathcal{M}$, respectively, because $\mathcal{F}$ and φ are supposed to be precompact. Thus $\mathcal{F}_0^{\mathfrak{P}_0} \in \mathcal{M}_{\mathbf{X},\mathbf{Y}}$ and $\varphi_0^{\mathfrak{P}_0} \in \mathcal{M}$, implying $\mathcal{F}_0^{\mathfrak{P}_0}(\varphi^{\mathfrak{P}_0}) \in \mathcal{N}$. But we have naturally $(\mathcal{F}_0(\varphi_0))^{\mathfrak{P}_0} \preceq \mathcal{F}_0^{\mathfrak{P}_0}(\varphi^{\mathfrak{P}_0})$, because for all $G \in \mathcal{F}_0, P \in \varphi_0$ we find $G(P) \in \{G'(P') | G' \in \mathfrak{P}_0(G), P' \in \mathfrak{P}_0(P)\}$ and consequently $\mathfrak{P}_0(G(P)) \preceq \{G'(P') | G' \in \mathfrak{P}_0(G), P' \in \mathfrak{P}_0(P)\}$. Now, by the refinement-closedness of $(Y, \mathcal{N})$ we get $(\mathcal{F}_0(\varphi_0))^{\mathfrak{P}_0} \in \mathcal{N}$, thus $\mathcal{F}_0(\varphi_0)$ is Cauchy, and consequently ψ is Cauchy, too, by proposition 75. Thus, $\mathcal{F}(\varphi)$ is precompact. ■

98 Corollary
Let $(X, \mathcal{M}) \in |\mathbf{PFS}|$ and $(Y, \mathcal{N})$ be a pseudoprincipal and refinement-closed power-filter-space. Then for each precompact subset $\mathcal{H} \subseteq \mathbf{Y}^{\mathbf{X}}$ hold:

(1) $\mathcal{H}$ is equiuniformly fine and

(2) For every $x \in X$ is
$\mathcal{H}(x) := \{f(x) | f \in \mathcal{H}\}$
a precompact subset of $(Y, \mathcal{N})$.

Proof: Apply lemma 97 to the principal filters $\mathcal{F} := [\mathcal{H}]$ and $\varphi := \overset{\bullet}{x}$. ■

Note, that the foregoing two statements hold for arbitrary multifilter-spaces $(X, \mathcal{M})$ and pseudoprincipal multifilter-spaces $(Y, \mathcal{N})$, too, because of lemma 59.

99 Proposition
Let $(X, \mathcal{M})$ be a limited multifilter-space, $(Y, \mathcal{N})$ a uniform principal multifilter-space, $P \in \mathcal{PC}(X)$ and $\mathcal{H} \subseteq Y^X$ a equiuniformly fine family with the property, that $\mathcal{H}(x) := \{h(x) | h \in \mathcal{H}\}$ is precompact for every $x \in P$. Then $\mathcal{H}(P) := \{h(p) | h \in \mathcal{H}, p \in P\}$ is precompact, too.

Proof: Let $\mathcal{N} := [\Xi]$ and let $\psi \in \mathfrak{F}_0(\mathcal{H}(P))$. For every $y \in \mathcal{H}(P)$ there exist $h_y \in \mathcal{H}, p_y \in P$ s.t. $y = h_y(p_y)$, thus (at least one) map $\pi : \mathcal{H}(P) \to P \times \mathcal{H}$ exists with $\pi(y) := (p_y, h_y)$. Then let π_1, π_2 be the canonical projections from $P \times \mathcal{H}$ to $P, \mathcal{H}$, respectively. Now, $\mathcal{F} := \pi_2(\pi(\psi))$ is an ultrafilter on $\mathcal{H}$, and $\chi := \pi_1(\pi(\psi))$ is an ultrafilter on P and therefore χ is Cauchy, i.e. $\exists \Sigma \in \mathcal{M} : \forall \sigma \in \Sigma : \exists S \in \sigma : S \in \chi$. Furthermore, we have $[\mathcal{H}^{\{\}}](\Sigma) \preceq \Xi$, because $\mathcal{H}$ is equiuniformly fine. So, let $\xi \in \Xi$ be given, then exists $\sigma \in \Sigma$ with $\mathcal{H}^{\{\}}(\sigma) \preceq \xi$. Now, we have $\emptyset \neq S \in \sigma$ with $S \in \chi$, so let $s \in S$. Then $\mathcal{F}(s)$ is an ultrafilter on $\mathcal{H}(s)$ and therefore Cauchy, by assumption. Thus, there exists $K \in \xi$ with $K \in \mathcal{F}(s)$. This yields $\mathcal{H}_1 := \{h \in \mathcal{H} | h(S) \cap K \neq \emptyset\} \in \mathcal{F}$, but because of our choice for σ, we have

$\forall h \in \mathcal{H} : \exists K_h \in \xi : h(S) \subseteq K_h$. But now $K \cup \bigcup_{h\in\mathcal{H}_1} K_h \in \xi^{*2}$ holds. This follows for all $\xi \in \Xi$, thus $\mathcal{F}(\chi)$ is Cauchy w.r.t. $\Xi^{*2} = \Xi$ and it is clearly coarser than ψ, thus ψ is Cauchy. ■

There are additional important multifilter- (resp. powerfilter-)structures for the set of functions between multifilter (resp. powerfilter-) spaces $(X, \mathcal{M}), (Y, \mathcal{N})$:

100 Definition
Let $(X, \mathcal{M}), (Y, \mathcal{N})$ be multifilter- (resp. powerfilter-) spaces. The multifilter- (resp. powerfilter-) structure

$$\mathcal{M}_{Y,p} := \{\Gamma \in \kappa |\ \forall x \in X : \Gamma(x) \in \mathcal{N}\}$$

is called the **pointwise** *multifilter- (resp. powerfilter-) structure on Y^X, where κ stands for the set of multifilters or the powerfilters, respectively, on Y^X and $\Gamma(x)$ is the multifilter (resp. powerfilter) on Y, generated from*
$\{\{\{g(x) |\ g \in G\} |\ G \in \gamma\} |\ \gamma \in \Gamma\}$.
The multifilter- (resp. powerfilter-) structure

$$\mathcal{M}_{Y,pc} := \{\Gamma \in \kappa |\ \forall \Sigma \in \mathcal{M} : \exists P \in \mathcal{PC}(X) \cap \Sigma^{\cup} \Rightarrow \Gamma(\Sigma) \in \mathcal{N}\}$$

is called the **precompactly fine** *structure.*

In both cases, it is trivial to check, that they are indeed multifilter- (resp. powerfilter-) structures. Note, that $\mathcal{M}_{Y,p}$ is just the product structure, if Y^X is identified in the usual manner with $\prod_{x\in X} Y_x$, where all Y_x are clones of Y.

101 Proposition
Let $(X, \mathcal{M}), (Y, \mathcal{N})$ be multifilter- (resp. powerfilter-) spaces. Then holds

$$\mathcal{M}_{Y,p} \supseteq \mathcal{M}_{Y,pc} \supseteq \mathcal{M}_{\mathbf{X},\mathbf{Y}} \ .$$

If $(X, \mathcal{M})$ is locally precompact, then $\mathcal{M}_{Y,pc} = \mathcal{M}_{\mathbf{X},\mathbf{Y}}$ holds.

Proof: $\Gamma \in \mathcal{M}_{\mathbf{X},\mathbf{Y}}$ just means $\Gamma(\Sigma) \in \mathcal{N}$ for all members of $\mathcal{M}$, so, this holds especially for the refinements $[\Sigma_{|P}] \preceq \Sigma \in \mathcal{M}$, and consequently $\Gamma \in \mathcal{M}_{Y,pc}$ follows, implying $\forall x \in X : \Gamma(x) \in \mathcal{N}$, because all singleton-multifilters $\widehat{x}$ (resp. powerfilters $[\{\{x\}\}]$) are of this type, with the singleton multifilter (resp. powerfilter) itself as Σ and $\{x\}$ as P, thus $\Gamma \in \mathcal{M}_{Y,p}$ follows. In case of locally precompactness for $(X, \mathcal{M})$, we get $\forall \Sigma \in \mathcal{M} : \mathcal{PC}(X) \cap \Sigma^{\cup} \neq \emptyset$ directly from definition 78, thus $\mathcal{M}_{Y,pc} \subseteq \mathcal{M}_{\mathbf{X},\mathbf{Y}}$ follows from the definition of $\mathcal{M}_{Y,pc}$. ■

4 Hyperspaces

4.1 Some Hyperstructures from Topological Spaces

Let (X,τ) be a topological space. By $Cl(X)$ and $K(X)$ we denote the family of all closed subsets and the set of all compact subsets of X, respectively. For $B \in \mathfrak{P}(X)$ and $\mathfrak{A} \subseteq \mathfrak{P}(X)$ we define $B^{-\mathfrak{A}} := \{A \in \mathfrak{A} | A \cap B \neq \emptyset\}$ (hit–set) and $B^{+\mathfrak{A}} := \{A \in \mathfrak{A} | A \cap B = \emptyset\}$ (miss–set). Specializing $\mathfrak{A} := Cl(X)$, we get the usual symbols B^-, B^+. By $\tau_{l,\mathfrak{A}}$ we denote the topology for $\mathfrak{A}$, generated by the subbase of all $G^{-\mathfrak{A}}, G \in \tau$. Now consider $\emptyset \neq \alpha \subseteq \mathfrak{P}(X)$; by $\tau_{\alpha,\mathfrak{A}}$ we denote the topology for $\mathfrak{A}$ which is generated from the subbase of all $B^{+\mathfrak{A}}, B \in \alpha$ and $G^{-\mathfrak{A}}, G \in \tau$. Of course, for every possible α we have $\tau_{l,\mathfrak{A}} \subseteq \tau_{\alpha,\mathfrak{A}}$; for $\alpha = Cl(X)$ we get the Vietoris topology and for $\alpha = K(X)$ we get the Fell topology for $\mathfrak{A}$. If $\alpha = \Delta \subseteq Cl(X)$, $\tau_{\alpha,\mathfrak{A}}$ is called Δ–topology by Beer and Tamaki [5].

4.1.1 Compactness Properties for Hit-and-Miss Topologies

102 Definition
If X is a set, $\tau, \mathfrak{A}$ are subsets of $\mathfrak{P}(X)$, then we call $\mathfrak{A}$ **weakly complementary w.r.t** *τ, iff for every subset $\sigma \subseteq \tau$ there exists a subset $\mathfrak{B} \subseteq \mathfrak{A}$, s.t. $\bigcup_{B\in\mathfrak{B}} B = X \setminus \bigcup_{S\in\sigma} S$.*

103 Lemma
(Covering Equivalence)
Let X be a set, $\tau, \mathfrak{A} \subseteq \mathfrak{P}(X)$ and $K \subseteq X$. Then holds

$$\bigcup_{i\in I} G_i \supseteq K \implies \bigcup_{i\in I} G_i^{-\mathfrak{A}} \supseteq K^{-\mathfrak{A}}$$

for every collection $G_i, i \in I, G_i \in \tau$.
If $\mathfrak{A}$ is weakly complementary w.r.t. τ, then for every collection $G_i, i \in I, G_i \in \tau$ the implication

$$\bigcup_{i\in I} G_i \supseteq K \impliedby \bigcup_{i\in I} G_i^{-\mathfrak{A}} \supseteq K^{-\mathfrak{A}}$$

holds, too.

Proof: Let $\bigcup_{i\in I} G_i \supseteq K$. $A \in K^{-\mathfrak{A}} \Rightarrow A \cap K \neq \emptyset \Rightarrow \emptyset \neq A \cap \bigcup_{i\in I} G_i \Rightarrow \exists i_0 \in I : A \cap G_{i_0} \neq \emptyset \Rightarrow A \in G_{i_0}^{-\mathfrak{A}} \Rightarrow A \in \bigcup_{i\in I} G_i^{-\mathfrak{A}}$.
Conversely, let $\mathfrak{A}$ be weakly complementary w.r.t. τ and $\bigcup_{i\in I} G_i^{-\mathfrak{A}} \supseteq K^{-\mathfrak{A}}$. Assume $\bigcup_{i\in I} G_i \not\supseteq K$. Then $X \setminus \bigcup_{i\in I} G_i \supseteq K \setminus \bigcup_{i\in I} G_i \neq \emptyset$ holds, so there is an $A \in \mathfrak{A}, A \subseteq X \setminus \bigcup_{i\in I} G_i$ with $A \cap K \setminus \bigcup_{i\in I} G_i \neq \emptyset$. Thus $A \in K^{-\mathfrak{A}}$, implying $A \in \bigcup_{i\in I} G_i^{-\mathfrak{A}}$. This yields $\exists i_0 \in I : A \cap G_{i_0} \neq \emptyset$ in contradiction to the construction of A. ∎

104 Corollary
Let X be a set, $\tau, \mathfrak{A} \subseteq \mathfrak{P}(X)$ and $K \subseteq X$. Then holds

$$\bigcup_{i \in I} G_i \supseteq K \iff \bigcup_{i \in I} G_i^{-\mathfrak{A}} \supseteq K^{-\mathfrak{A}} \tag{3}$$

for every collection $G_i, i \in I, G_i \in \tau$ if and only if $\mathfrak{A}$ is weakly complementary w.r.t. τ.

Proof: We only have to show, that $\mathfrak{A}$ is weak complementary w.r.t. τ, if (3) holds. Assume, $\mathfrak{A}$ is not weakly complementary w.r.t. τ. Then there must be a collection $\{G_i | i \in I\} \subseteq \tau$, such that $\bigcup\{A | A \in \mathfrak{P}(X \setminus \bigcup_{i \in I} G_i) \cap \mathfrak{A}\} \not\supseteq X \setminus \bigcup_{i \in I} G_i$. Now, we chose $K := \left(X \setminus \bigcup_{i \in I} G_i\right) \setminus \bigcup\{A | A \in \mathfrak{P}(X \setminus \bigcup_{i \in I} G_i) \cap \mathfrak{A}\} \neq \emptyset$. Then no element of $\mathfrak{A}$, which meets K, can be contained in $X \setminus \bigcup_{i \in I} G_i$, i.e. every element of $K^{-\mathfrak{A}}$ meets $\bigcup_{i \in I} G_i$, too. So, it must meet a $G_{i_0}, i_0 \in I$ and consequently it is contained in $\bigcup_{i \in I} G_i^{-\mathfrak{A}}$. But, by construction, the collection $\{G_i | i \in I\}$ doesn't cover K, so (3) would fail. ∎

Obviously, if for every collection $\{G_i | i \in I\} \subseteq \tau$ the complement $X \setminus \bigcup_{i \in I} G_i$ itself belongs to $\mathfrak{A}$, or if all singletons $\{x\}, x \in X$ are elements of $\mathfrak{A}$, then $\mathfrak{A}$ is weakly complementary w.r.t. τ. So, if τ is a topology on X, $Cl(X)$ and $K(X)$ are weakly complementary w.r.t. τ.

105 Corollary
Let (X, τ) be a topological space, $K \subseteq X$ and $\forall i \in I : G_i \in \tau$. Then holds

$$\bigcup_{i \in I} G_i \supseteq K \iff \bigcup_{i \in I} G_i^{-} \supseteq K^{-}$$

106 Definition
Let κ be a cardinal. Then a topological space (X, τ) is called **κ-compact**, *iff every open cover of X with cardinality at most κ admits a finite subcover.*
(X, τ) is called **κ-Lindelöf**, *iff every open cover of X admits a subcover of cardinality at most κ.*[6]

A filter is called κ-generated, iff it has a base of cardinality at most κ. A filter φ is called κ-completable, iff every subset $\mathfrak{B} \subseteq \varphi$ with $card(\mathfrak{B})$ at most κ fulfills $\bigcap_{B \in \mathfrak{B}} B \neq \emptyset$. It is called κ-complete, iff $\bigcap_{B \in \mathfrak{B}} B \in \varphi$ holds under these conditions.

107 Proposition
A topological space (X, τ) is κ-compact, if and only if every κ-generated filter on X has a convergent refining ultrafilter.

[6]These notions are defined a little different than elsewhere, as in [10] for instance. However, in our opinion, the notions chosen here, seem to be more consistent with the quite familiar notion of countable compactness.

Proof: Let (X,τ) be κ-compact and φ a filter on X with a base $\mathfrak{B}$ of cardinality at most κ. Assume, all refining ultrafilters of φ would fail to converge in X. Then for each element $x \in X$ holds, that all refining ultrafilters of φ contain the complement of an open neighbourhood of x. But the set of complements of open neighbourhoods of a point x is closed w.r.t. finite unions, thus by lemma 9 φ contains the complement of an open neigbourhood of x. So, for each $x \in X$ there must exist $O_x \in \tau \cap \overset{\bullet}{x}$ and $B_x \in \mathfrak{B}$, s.t. $B_x \subseteq X \setminus O_x$, implying $\overline{B_x} \subseteq X \setminus O_x$ and thus $X \setminus \overline{B_x} \supseteq O_x$. Now, for each $B \in \mathfrak{B}$ we define $O_B := X \setminus \overline{B}$ and find, that $\{O_B |\ B \in \mathfrak{B}\}$ is an open cover of X, because of the preceding facts. So, there must exist a finite subcover $O_{B_1} \cup \cdots \cup O_{B_n} = X$, implying $\bigcup_{i=1}^{n}(X \setminus \overline{B_i}) = X$, just meaning $\bigcup_{i=1}^{n} \overline{B_i} = \emptyset$, which is impossible, because all B_i belong to the filter φ. So, the assumption must be false; there must exist convergent refining ultrafilters of φ.
Otherwise, let all κ-generated filter on X have a convergent refining ultrafilter. Assume, there would exist an open cover $\mathfrak{C} := \{O_i \in \tau |\ i \in I\}, \bigcup_{i\in I} O_i = X, card(I) \leq \kappa$ such that all finite subcollections fail to cover X (implying κ to be infinite). But the set of all finite subcollections of the infinite collection $\mathfrak{C}$ of cardinality at most κ has cardinality at most κ, too. So, $\mathfrak{B} := \{X \setminus \bigcup_{k=1}^{n} O_{i_k} |\ n \in I\!N, i_k \in I\}$ is a filterbasis of cardinality at most κ, thus there must exist an ultrafilter $\varphi \supseteq \mathfrak{B}$, which converges in X - leading to the usual contradiction, because every $x \in X$ is contained in an open $O_x \in \mathfrak{C}$ and $X \setminus O_x$ belongs to $\mathfrak{B} \subseteq \varphi$. ■

Analogously we get a characterization of κ-Lindelöf-spaces.

108 Proposition
If (X,τ) is κ-Lindelöf, then every κ-completable filter on X has a convergent refining ultrafilter.
If κ is an infinite cardinal and every κ-complete filter on a topological space (X,τ) has a convergent refining ultrafilter, then (X,τ) is κ-Lindelöf.

Proof: Let (X,τ) be κ-Lindelöf and $\varphi \in \mathfrak{F}(X)$ κ-completable.
Assuming all refining ultrafilters of φ to be non-convergent, we get in the same way as before for every $x \in X$ an $O_x \in \tau \cap \overset{\bullet}{x}$ s.t. $X \setminus O_x \in \varphi$. These $O_x, x \in X$ form an open cover of X, which must contain a subcover of cardinality at most κ. But $\bigcup_{i\in I} O_{x_i} = X$ with $card(I) \leq \kappa$ just means $\bigcap_{i\in I}(X \setminus O_{x_i}) = \emptyset$ - in contradiction to the κ-completability of φ.
Now, let every κ-complete Filter on X have a convergent refining ultrafilter.
Let $\{O_i | i \in I, O_i \in \tau\}$ be given with $\bigcup_{i\in I} O_i = X$. Assume $\forall J \subseteq I, card(J) \leq \kappa :$ $X \setminus \bigcup_{j\in J} O_j \neq \emptyset$. Then $\mathfrak{B} := \{X \setminus \bigcup_{j\in J} O_j |\ J \subseteq I, card(J) \leq \kappa\}$ is a base for a κ-complete filter, because every union of at most κ sets of cardinality at most κ has cardinality at most κ, too. So, there must be an ultrafilter $\psi \supseteq \mathfrak{B}$, which converges in X - yielding the usual contradiction, because $\mathfrak{B}$ contains the complement of an open neigbourhood for each $x \in X$. ■

Of course, every κ-complete filter is κ-completable, so we may say, that a topological space (X,τ) is κ-Lindelöf, if and only if each κ-complete filter on X has a convergent refinement.

109 Lemma
Let κ be a cardinal, (X,τ) a topological space and let $\mathfrak{A} \subseteq \mathfrak{P}(X)$ be weakly complementary w.r.t. τ. If $\mathfrak{A}_0 := \mathfrak{A} \setminus \{\emptyset\}$ is κ-Lindelöf (resp. κ-compact) in $\tau_{l,\mathfrak{A}_0}$, then (X,τ) is κ-Lindelöf (resp. κ-compact).

Proof: If $\mathfrak{A}$ is weakly complementary w.r.t. τ, then $\mathfrak{A}_0$ is, too. So, corollary 104 is applicable. Let $\{G_i | i \in I\}$ be an open cover (resp. an open cover with cardinality at most κ) of X. By corollary 104, then $\{G_i^{-\mathfrak{A}_0} | i \in I\}$ is an open cover of $X^{-\mathfrak{A}_0} = \mathfrak{A}_0$ (resp. of card. at most κ), so there exists a subset $J \subseteq I$ of cardinality at most κ (resp. a finite subset J), s.t. $\bigcup_{j\in J} G_j^{-\mathfrak{A}_0} \supseteq \mathfrak{A}_0 = X^{-\mathfrak{A}_0}$, implying $\bigcup_{j\in J} G_j \supseteq X$ by corollary 104. ■

Of course, the assumed topology $\tau_{l,\mathfrak{A}_0}$ is not really hit-and-miss, because the miss-sets are missed. But every proper hit-and-miss-topology would be stronger and therefore it would enforce the desired properties for (X,τ) as well.

110 Corollary
Let (X,τ) be a topological space and let $\mathfrak{A} \subseteq \mathfrak{P}(X)$ be weakly complementary w.r.t. τ. If $\mathfrak{A}_0 := \mathfrak{A} \setminus \{\emptyset\}$ is compact in $\tau_{l,\mathfrak{A}_0}$, then (X,τ) is compact.

111 Lemma
Let (X,τ) be a κ-compact (resp. κ-Lindelöf) topological space and assume $Cl(X) \subseteq \mathfrak{A} \subseteq \mathfrak{P}(X)$. Then $\mathfrak{A}_0 := \mathfrak{A} \setminus \{\emptyset\}$ is κ-compact (resp. κ-Lindelöf) in $\tau_{l,\mathfrak{A}_0}$.

Proof: Let $\hat{\varphi}$ be a κ-generated (resp. κ-complete) filter on $\mathfrak{A}_0$. Then, for an arbitrary $h \in \mathcal{A} := \{g \in X^{\mathfrak{P}_0(X)} | \ \forall M \in \mathfrak{P}_0(X) : g(M) \in M\}$ the image $h(\hat{\varphi})$ is a κ-generated (resp. κ-complete) filter on X and consequently it has a τ-convergent refining ultrafilter ψ_h. Furthermore, there must exist an ultrafilter $\hat{\psi} \supseteq \hat{\varphi}$, s.t. $h(\hat{\psi}) = \psi_h$. So, the set

$$A := \{a \in X | \ \exists f \in \mathcal{A} : (f(\hat{\psi}), a) \in q_\tau\}$$

is not empty and consequently the closure $\overline{A}$ belongs to $\mathfrak{A}_0$. Now, for any $O \in \tau$ with $\overline{A} \in O^{-\mathfrak{A}_0}$ ($\Leftrightarrow \overline{A} \cap O \neq \emptyset$) we get $A \cap O \neq \emptyset$ (because of the closure-properties). Now, the assumption $O^{-\mathfrak{A}_0} \notin \hat{\psi}$ would imply $O^{+\mathfrak{A}_0} \in \hat{\psi}$, yielding $\forall f \in \mathcal{A} : X \setminus O \in f(\hat{\psi})$, thus $\forall f \in \mathcal{A} : \forall b \in A \cap O : (f(\hat{\psi}), b) \notin q_\tau$ - in contradiction to the construction of A. Thus, $O \in \tau, \overline{A} \in O^{-\mathfrak{A}_0}$ always imply $O^{-\mathfrak{A}_0} \in \hat{\psi}$ and consequently $\hat{\psi}$ $\tau_{l,\mathfrak{A}_0}$-converges to $\overline{A}$. ■

112 Corollary
Let (X,τ) be a compact topological space and assume $Cl(X) \subseteq \mathfrak{A} \subseteq \mathfrak{P}(X)$. Then $\mathfrak{A}_0 := \mathfrak{A} \setminus \{\emptyset\}$ is compact in $\tau_{l,\mathfrak{A}_0}$.

113 Definition
Let (X,τ) be a topological space. A subset $A \subseteq X$ is called bf weak relative complete in X, iff

$$\forall \varphi \in \mathfrak{F}(A) \cap q_\tau^{-1}(X) : \mathfrak{F}(\varphi) \cap q_\tau^{-1}(A) \neq \emptyset \ ,$$

i.e. every filter φ on A, which converges in X, has a refinement, converging in A.

114 Proposition
Let (X,τ) be a topological space and $A \subseteq X$. Then holds:

(1) A is weak relative complete in X, iff $\mathfrak{F}_0(A) \cap q_\tau^{-1}(X) = \mathfrak{F}_0(A) \cap q_\tau^{-1}(A)$, i.e. every ultrafilter on A, which converges in X, converges in A, too.

(2) If A is closed in X, then A is weak relative complete in X.

(3) If A is compact, then A is weak relative complete in X.

(4) If (X,τ) is compact and A is weak relative complete in X, then A is compact, too.

(5) If (X,τ) is Hausdorff, then every weak relative complete subset $A \subseteq X$ is closed in (X,τ).

(6) A is compact iff A is weak relative complete and relative compact.

(7) If (X,τ) is κ-compact and A is weak relative complete in (X,τ), then A is κ-compact.

(8) If (X,τ) is κ-Lindelöf and A is weak relative complete in (X,τ), then A is κ-Lindelöf.

(9) Weak relative completeness is transitive, i.e. for all $A \subseteq B \subseteq X$ with B weak relative complete in (X,τ) and A weak relative complete in $(B,\tau_{|B})$, the subset A is weak relative complete in (X,τ), too.

Proof: (1): If A is weak relative complete in X, the assertion about the ultrafilters on A follows immediately from the fact, that an ultrafilter has no proper refinement. Conversely, if a filter φ on A is given, which converges in X, then every refining ultrafilter $\psi \supseteq \varphi$ converges in X, too. Now, by $\mathfrak{F}_0(A) \cap q_\tau^{-1}(X) = \mathfrak{F}_0(A) \cap q_\tau^{-1}(A)$, ψ converges in A and is a refinement of φ. So, A is weak relative complete in X.
(2): If A is closed in X, then every point of X, to which a filter on A may converge, belongs to A.
(3): If A is compact, then every ultrafilter on A converges in A and the weak relative completeness of A in X follows from (1).
(4): X compact $\Rightarrow \mathfrak{F}_0(X) \cap q_\tau^{-1}(X) = \mathfrak{F}_0(X) \Rightarrow \mathfrak{F}_0(A) \cap q_\tau^{-1}(X) = \mathfrak{F}_0(A)$ and by the weak relative completeness of A with (1) we get $\mathfrak{F}_0(A) \cap q_\tau^{-1}(A) = \mathfrak{F}_0(A)$, i.e. A

is compact.
(5): If A is weak relative complete in (X,τ) and there is a filter $\varphi \in \mathfrak{F}(A)$, converging to a point $x \in X$. Then there must exist a refining filter $\psi \in \mathfrak{F}(\varphi)$ which converges to a point $a \in A$. But this filter converges to x, too, because of it's subfilter φ, so by Hausdorffness $x = a \in A$ follows. So, A is closed in (X,τ).
(6): A compact subset A is clearly relative compact, and it is weak relative complete by (3). If A is relative compact, then every ultrafilter on A converges in X and so it converges in A by (1), if additionally A is weak relative complete in X.
(7): follows directly from (1) and proposition 107.
(8): follows directly from (1) and proposition 108.
(9): Follows immediately from (1), because an ultrafilter on A is an ultrafilter on B, too. So, if it converges in X, it must converge in B and so in A, too, because of the weak relative completeness, successively. ■

The idea may occur, that every weak relative complete subset of a topological space could be closed or compact, but this is not the case: let $X := \mathbb{R} \cup \{i\}$, τ_e the euclidian topology on $\mathbb{R}$ and $\tau := \tau_e \cup \{O \cup \{i\} |\ O \in \overset{\bullet}{0} \cap \tau_e\}$, then $(0,\infty) \cup \{i\}$ is weak relative complete in (X,τ), but neither closed nor compact.
There is also a description by coverings for weak relative completeness.

115 Lemma
Let (X,τ) be a topological space and $A \subseteq X$. Then the following are equivalent:

(1) A is weak relative complete in X.

(2) For every open cover $\mathfrak{A}$ of A and every element x of X, there is an open neighbourhood $U_{x,\mathfrak{A}}$ of x, s.t. $U_{x,\mathfrak{A}} \cap A$ is covered by finitely many members of $\mathfrak{A}$.

(3) For every open cover $\mathfrak{A}$ of A exists an open cover $\mathfrak{A}' \supseteq \mathfrak{A}$ of X, such that the intersection of every member of $\mathfrak{A}'$ with A can be covered by finitely many members of $\mathfrak{A}$, i.e. $\forall O \in \mathfrak{A}' : \exists n \in \mathbb{N}, P_1, ..., P_n \in \mathfrak{A} : \bigcup_{i=1}^{n} P_i \supseteq O \cap A$ holds.

Proof: (1)⇒(2): Let $\mathfrak{A} \subseteq \tau$ with $\bigcup_{P \in \mathfrak{A}} P \supseteq A$ be given. For every $x \in A$ we can chose a single member of $\mathfrak{A}$ as open neighbourhood, whose intersection with A is covered by itself. So, assume

$$\exists x \in X \setminus A : \forall U_x \in \underline{U}(x) \cap \tau : \forall n \in \mathbb{N}, P_1, ..., P_n \in \mathfrak{A} : U_x \cap A \not\subseteq \bigcup_{i=1}^{n} P_i \qquad (4)$$

Then $\mathfrak{B} := \{(U \cap A) \setminus \bigcup_{i=1}^{n} P_i |\ U \in \underline{U}(x) \cap \tau, n \in \mathbb{N}, P_i \in \mathfrak{A}\}$ would be closed under finite intersections and thus there would exist an ultrafilter φ on A with $\varphi \supseteq \mathfrak{B}$. By construction $\varphi \to x$ must hold for this ultrafilter, and now by the weak relative completeness of A it follows $\exists a \in A : \underline{U}(a) \subseteq \varphi$. But $\mathfrak{A}$ is an open cover of A, so

there is an open set $P \in \mathfrak{A}$ with $a \in P$, implying $P \in \varphi$ – in contradiction to the construction of φ. Thus (4) is false and we have

$$\forall x \in X \setminus A : \exists U_x \in \underline{U}(x) \cap \tau : \exists n \in I\!N, P_1, ..., P_n \in \mathfrak{A} : U_x \cap A \subseteq \bigcup_{i=1}^{n} P_i$$

(2)⇒(3): Note, that (3) is fulfilled with $\mathfrak{A}' := \{U_x |\ x \in X \setminus A\} \cup \mathfrak{A}$.
(3)⇒(1): For a given ultrafilter φ on A with $\varphi \to x \in X$ assume $\varphi \notin q_\tau^{-1}(A)$. Then $\forall a \in A : \exists U_a \in \underline{U}(a) \cap \tau : U_a^c = X \setminus U_a \in \varphi$. With these neighborhoods define $\mathfrak{A} := \{U_a |\ a \in A\}$, which is an open cover of A. By (2) there is an open cover $\mathfrak{A}' \supseteq \mathfrak{A}$ of X such that $\forall O \in \mathfrak{A}' : \exists n \in I\!N, P_1, ..., P_n \in \mathfrak{A} : \bigcup_{i=1}^{n} P_i \supseteq O \cap A$ holds. Now, $\varphi \to x$ implies $\exists O \in \mathfrak{A}' : O \in \varphi$ (especially $A \cap O \neq \emptyset$ follows), and then we have $\exists n \in I\!N, P_1, ..., P_n \in \mathfrak{A} : O \cap A \subseteq \bigcup_{i=1}^{n} P_i$, implying $\exists j \in \{1, ..., n\} : P_j \in \varphi$ – in contradiction to the construction of $\mathfrak{A}$. So, the assumption $\varphi \notin q_\tau^{-1}(A)$ must be false, showing, that every ultrafilter on A, which converges in X, converges in A, too. ■

116 Theorem
Let (X, τ) be a topological space, and let $\alpha \subseteq \mathfrak{P}(X)$ consist of weakly relative complete subsets of X. Then holds for any $\mathfrak{A}$ with $Cl(X) \subseteq \mathfrak{A} \subseteq \mathfrak{P}(X)$:
$(\mathfrak{A}_0, \tau_\alpha)$ is compact $\Longleftrightarrow$ (X, τ) is compact.

Proof: According to lemma 109 we only must show that $(\mathfrak{A}_0, \tau_\alpha)$ is compact, if (X, τ) is compact. So, assuming (X, τ) to be compact, by proposition 114 every weakly relative complete subset of X is compact, too, and we have $\alpha \subseteq K(X)$. Now we will use Alexander's lemma: let $\underline{U}$ be a cover of $\mathfrak{A}_0$, consisting of subbase elements $K_i^{+\mathfrak{A}_0}, G_j^{-\mathfrak{A}_0}$ with K_i compact and G_j open.
$A := X \setminus (\bigcup\{G | G^{-\mathfrak{A}_0} \in \underline{U}\})$ is closed.
By construction, $A \notin G^{-\mathfrak{A}_0}$ for any $G^{-\mathfrak{A}_0} \in \underline{U}$, so for $A \neq \emptyset$ there must exist some $K_0^{+\mathfrak{A}_0} \in \underline{U}$ with $A \in K_0^{+\mathfrak{A}_0}$, yielding that $K_0 \subseteq \bigcup\{G | G^{-\mathfrak{A}_0} \in \underline{U}\}$; K_0 compact ⇒ $\exists G_1, ..., G_n \in \underline{U}$ with $K_0 \subseteq \bigcup_{k=1}^{n} G_k$, but then $\{K_0^{+\mathfrak{A}_0}\} \cup \{G_1^{-\mathfrak{A}_0}, ..., G_n^{-\mathfrak{A}_0}\}$ is a cover of $\mathfrak{A}_0$.
If $A = \emptyset$, then $\bigcup\{G_i | G_i^{-\mathfrak{A}_0} \in \underline{U}\} = X$, so from the compactness of X the existence of some $G_1^{-\mathfrak{A}_0}, ..., G_n^{-\mathfrak{A}_0} \in \underline{U}$ with $X = \bigcup_{k=1}^{n} G_k$ follows. By lemma 103 then $\bigcup_{k=1}^{n} G_k^{-\mathfrak{A}_0} = \mathfrak{A}_0$ holds. ■

Most of the well-known theorems for compactness w.r.t. the Fell– or the Vietoris–topology follow immediately from this.

117 Lemma
Let (X, τ) be a topological space, $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ with $Cl(X) \subseteq \mathfrak{A}$ and $\alpha \subseteq Cl(X)$. If $R \subseteq X$ is relative compact in X, then $\mathfrak{P}_0(R) \cap \mathfrak{A}$ is relative compact in $(\mathfrak{A}, \tau_\alpha)$.

Proof: Let $\mathfrak{B} := \{O_i^{-\mathfrak{a}} |\ i \in I, O_i \in \tau\} \cup \{C_j^{+\mathfrak{a}} |\ j \in J, C_j \in \alpha\}$ be an open cover of $\mathfrak{A}$ by subbase elements of τ_α. Let $O := \bigcup_{i\in I} O_i$.
If $O = X$, then there exists finitely many $i_1, ..., i_n \in I$ with $\bigcup_{k=1}^n O_{i_k} \supseteq R$, because R is relative compact, and thus $\bigcup_{k=1}^n O_{i_k}^{-\mathfrak{a}} \supseteq R^{-\mathfrak{a}} \supseteq \mathfrak{P}_0(R) \cap \mathfrak{A}$, by lemma 103.
If $O \neq X$, then $X \setminus O$ is nonempty and closed, but not covered by the $O_i^{-\mathfrak{a}}$ from $\mathfrak{B}$. Thus, there must exist a $j_0 \in J$ with $X \setminus O \in C_{j_0}^{+\mathfrak{a}}$, implying $C_{j_0} \subseteq O$. Now, we have $\mathfrak{P}_0(R) \cap \mathfrak{A} = (\mathfrak{P}_0(R) \cap C_{j_0}^{+\mathfrak{a}}) \cup (\mathfrak{P}_0(R) \cap C_{j_0}^{-\mathfrak{a}})$, and, of course, $\mathfrak{P}_0(R) \cap C_{j_0}^{+\mathfrak{a}}$ is covered just by $C_{j_0}^{+\mathfrak{a}} \in \mathfrak{B}$. So, we have to find a finite subcover for $(\mathfrak{P}_0(R) \cap C_{j_0}^{-\mathfrak{a}})$, if this is not empty. Observe, that $R \cap C_{j_0}$ is relative compact in X, because it is a subset of R. Furthermore, $\{O_i |\ i \in I\} \cup \{X \setminus C_{j_0}\}$ is an open cover of X. Thus we find again finitely many $i_1, .., i_n \in I$, s.t. $\bigcup_{k=1}^n O_{i_k} \supseteq R \cap C_{j_0}$ (because $X \setminus C_{j_0}$ can be removed from any cover of $R \cap C_{j_0}$ without to lose the covering property). Therefore $\bigcup_{k=1}^n O_{i_k}^{-\mathfrak{a}} \supseteq (R \cap C_{j_0})^{-\mathfrak{a}}$, by lemma 103. But $\mathfrak{P}_0(R) \cap C_{j_0}^{-\mathfrak{a}} \subseteq (R \cap C_{j_0})^{-\mathfrak{a}}$ holds, because any subset of R, which hits C_{j_0}, automatically hits $R \cap C_{j_0}$. ■

As an interesting application of an also quite simple set-theoretical property, concerning the $^+$-operator, we want to take a very short look on the naturally arising question, wheither a union of compact sets itself is compact. Michael showed in [24] that a union of closed sets is compact, if the unified family is compact w.r.t. the Vietoris-topology. Now, the Vietoris-topology is commonly induced by the upper-Vietoris τ_V^+(miss sets: $A^{+\mathfrak{a}}$ with $A^c \in \tau$) and τ_l, but τ_l is not sufficient to enforce compactness of a union of compact sets, as the following example shows: Let $X := I\!R$, endowed with euclidian topology, $\mathfrak{M} := \{[-m, m] |\ m \in I\!N\}$. Then $\bigcup_{M\in\mathfrak{M}} M = I\!R$, is obviously not compact. But every covering of $\mathfrak{M}$ with elements of the defining subbase for τ_l must especially cover the element $\{0\} = [0, 0]$ of $\mathfrak{M}$, so it must contain a set O^- with $0 \in O$. Now, every element of $\mathfrak{M}$ contains the point 0, too, thus $\mathfrak{M} \subseteq O^-$ follows. So, $\mathfrak{M}$ is compact in τ_l by Alexander's subbase lemma.
And unifying compact sets, τ_l is not necessary, too, as we will see.

118 Proposition
Let X be a set, $\mathfrak{X} \subseteq \mathfrak{P}(X)$ and $\mathfrak{M} \subseteq \mathfrak{X}$. Then holds

$$\bigcup_{i\in I} C_i^{+\mathfrak{x}} \supseteq \mathfrak{M} \Longrightarrow \bigcup_{i\in I} C_i^c \supseteq \bigcup_{M\in\mathfrak{M}} M$$

for every collection $C_i, i \in I$.

Proof: For every $M \in \mathfrak{M}$ there must exist an $i_M \in I$ with $M \in C_{i_M}^{+\mathfrak{x}}$, because of $\bigcup_{i\in I} C_i^{+\mathfrak{x}} \supseteq \mathfrak{M}$. Thus $M \subseteq C_{i_M}^c \subseteq \bigcup_{i\in I} C_i^c$. ■

In [20] was shown

119 Lemma
Let (X,τ) be a topological space and $\mathfrak{M} \subseteq K(X)$ compact w.r.t. the Vietoris topology. Then

$$K := \bigcup_{M\in\mathfrak{M}} M$$

is compact w.r.t. τ.

In fact, it would be enough to require compactness of $\mathfrak{M}$ w.r.t. the *upper* Vietoris topology, here.

Applying our simple set-theoretical statement, we get a similar result for unions of relative compact subsets.

120 Lemma
Let (X,τ) be a topological space, let $\mathfrak{X}$ be the family of all relative compact subsets of X and let $\mathfrak{M} \subseteq \mathfrak{X}$ be relative compact in $\mathfrak{X}$ w.r.t. the upper Vietoris topology. Then

$$R := \bigcup_{M\in\mathfrak{M}} M$$

is relative compact in (X,τ).

Proof: Let $\bigcup_{i\in I} O_i \supseteq X$ with $O_i \in \tau, i \in I$ an open covering of X. Because of the relative compactness of all $P \in \mathfrak{X}$, there is a finite subcovering $O_{i_P^1}, ..., O_{i_P^{n_P}}$ for every $P \in \mathfrak{X}$, i.e. $O_P := \bigcup_{k=1}^{n_P} O_{i_P^k} \supseteq M$. Of course, $O_P \in \tau$ and so $(O_P)^c$ is closed w.r.t. τ. Furthermore, $P \cap O_P^c = \emptyset$, implying $P \in (O_P^c)^{+_{\mathfrak{X}}}$. Thus we have $\mathfrak{X} \subseteq \bigcup_{P\in\mathfrak{X}} (O_P^c)^{+_{\mathfrak{X}}}$, where the $(O_P^c)^{+_{\mathfrak{X}}}$ are just open w.r.t. the upper–Vietoris topology. Because of the relative compactness of $\mathfrak{X}$ w.r.t. the upper–Vietoris topology, there must exist finitely many $P_1, ..., P_n \in \mathfrak{X}$ with $\mathfrak{M} \subseteq \bigcup_{j=1}^{n} (O_{P_j}^c)^{+_{\mathfrak{X}}}$. Now, from proposition 118 we get $R = \bigcup_{M\in\mathfrak{M}} M \subseteq \bigcup_{j=1}^{n} O_{P_j}$, where every O_{P_j} is a finite union of members of the original covering $\{O_i | i \in I\}$ by construction. ■

121 Corollary
Let (X,τ) be a topological space and let $\mathfrak{M} \subseteq \mathfrak{P}_0(X)$ consist of relative compact subsets of X. If $\mathfrak{M}$ is compact w.r.t. the upper–Vietoris topology, then

$$R := \bigcup_{M\in\mathfrak{M}} M$$

is relative compact in (X,τ).

Proof: $\mathfrak{M}$ is compact and therefore relative compact in every set, which contains $\mathfrak{M}$, especially in the family of all relative compact subsets of X. So, lemma 120 applies.■

4.1.2 Extending Hyperstructures to Sets of Filters on the Base Space

We need a little more notation: for $\Phi \in \mathfrak{F}(\mathfrak{F}(X))$ we define

$$\Phi^{\uparrow} := \left\langle \left\{ \mathfrak{F}_0(\bigcap_{\chi\in\mathfrak{A}} \chi) \,\middle|\, \mathfrak{A} \in \Phi \right\} \right\rangle$$

which is a filter on $\mathfrak{F}_0(X)$. In case, that Φ is a filter on $\mathfrak{P}_0(X)$, we represent by the same symbol $\Phi^{\uparrow}$ just the filter on $\mathfrak{F}_0(X)$, which we get by mapping all nonempty subsets of X to their generated principal filters and then applying the $^{\uparrow}$-operator. Furthermore, for $\Phi \in \mathfrak{F}(\mathfrak{F}(X))$ we set

$$\Phi^{\cup\cap} := \bigcup_{\mathfrak{A}\in\Phi} \bigcap_{\varphi\in\mathfrak{A}} \varphi \ .$$

122 Proposition
Let X, Y be sets.

(1) If Φ is an ultrafilter on $\mathfrak{F}_0(X)$, then $\Phi^{\cup\cap}$ is an ultrafilter on X.

(2) If Φ is a filter on $\mathfrak{F}_0(X)$ and $f \in Y^X$, then $f(\Phi^{\cup\cap}) \subseteq f(\Phi)^{\cup\cap}$ holds.

Proof: (1): Let $A \in \mathfrak{P}(X)$. Then every ultrafilter on X either contains A or A^c. Thus $\mathfrak{F}_0(A) \cup \mathfrak{F}_0(A^c) = \mathfrak{F}_0(X)$, implying that either $\mathfrak{F}_0(A)$ or $\mathfrak{F}_0(A^c)$ is contained in Φ. But in the first case A and in the second case A^c belongs to $\Phi^{\cup\cap}$.
(2): From $A \in f(\Phi^{\cup\cap})$ we get $\exists \mathfrak{M} \in \Phi : A \in f(\bigcap_{\chi\in\mathfrak{M}} \chi)$ and we always have $f(\bigcap_{\chi\in\mathfrak{M}} \chi) \subseteq \bigcap_{\chi\in\mathfrak{M}} f(\chi)$, so we get $\exists \mathfrak{N}(:= f(\mathfrak{M})) \in f(\Phi) : A \in \bigcap_{\xi\in\mathfrak{N}} \xi$, just implying $A \in f(\Phi)^{\cup\cap}$. ■

123 Proposition
Let (X, τ) be a topological space, $\mathfrak{X} \subseteq \mathfrak{P}_0(X)$, $\Phi \in \mathfrak{F}(\mathfrak{X})$ and $A \in \mathfrak{X}$. Then for the upper Vietoris-topology τ_V^+ holds

$$(\Phi, A) \in q_{\tau_V^+} \Longleftrightarrow \forall \Phi_1 \in \mathfrak{F}_0(\Phi^{\uparrow}) : \exists \varphi \in \mathfrak{F}_0(A) : \Phi_1^{\cup\cap} \supseteq \varphi \cap \tau \ .$$

Proof: Assume, there would exist an $\Phi_1 \in \mathfrak{F}_0(\Phi^{\uparrow})$ s.t. $\forall \varphi \in \mathfrak{F}_0(A) : \exists U_\varphi \in \varphi \cap \tau : U_\varphi \notin \Phi_1^{\cup\cap}$, i.e. every ultrafilter φ on A contains a member of the family $\alpha := \tau \setminus \Phi_1^{\cup\cap}$. But this family is closed under finite unions because of proposition 7, so lemma 9 applies and we get $[A] \cap \alpha \neq \emptyset$, i.e. $\exists O \in \tau : A \subseteq O \wedge O \notin \Phi_1^{\cup\cap}$. This implies $O^c \in \Phi_1^{\cup\cap}$, because $\Phi_1^{\cup\cap}$ is an ultrafilter, leading to $\mathfrak{F}_0(O^c) \in \Phi_1$ from which $(O^c)^{+\mathfrak{x}} \notin \Phi$ follows, thus $(\Phi, A) \notin q_{\tau_V^+}$.
Otherwise, let $(\Phi, A) \notin q_{\tau_V^+}$ be given. Then there exists an $O \in \tau$ with $A \in O$ and $(O^c)^{+\mathfrak{x}} \notin \Phi$. This means $\mathfrak{A} \setminus (O^c)^{+\mathfrak{x}} \neq \emptyset$ for all $\mathfrak{A} \in \Phi$, implying that $\{\{\varphi_0 \in \mathfrak{F}_0(X) |\ \exists K \in \mathfrak{A} : \varphi_0 \in \mathfrak{F}_0(K \setminus O)\} |\ \mathfrak{A} \in \Phi\}$ is a base for a filter on

$\mathfrak{F}(X)$, which refines $\Phi^\uparrow$, and there must be an ultrafilter Φ_1, containing this base and therefore containing $\Phi^\uparrow$. But obviously, $O \notin \Phi_1^{\curlyvee}$, and so $\Phi_1^{\curlyvee}$ doesn't contain $\varphi \cap \tau$ for any ultrafilter φ on A. ■

124 Proposition
Let (X,τ) be a topological space, $\mathfrak{X} \subseteq \mathfrak{P}_0(X)$, $\Phi_0 \in \mathfrak{F}_0(\mathfrak{X})$ and $A \in \mathfrak{X}$. Then for the lower semifinite topology τ_l holds

$$(\Phi_0, A) \in q_{\tau_l} \iff \forall \varphi \in \mathfrak{F}_0(A) : \exists \Phi_1 \in \mathfrak{F}_0(\Phi_0^\uparrow) : \Phi_1^{\curlyvee} \supseteq \varphi \cap \tau \ .$$

Proof: Let $\Phi_0 \in \mathfrak{F}_0(\mathfrak{X}), A \in \mathfrak{X}, (\Phi_0, A) \in q_{\tau_l}$ and $\varphi \in \mathfrak{F}_0(A)$. Then $\forall U \in \varphi \cap \tau : U^- \in \Phi_0$, implying that $\mathfrak{B} := \{B_{U,\mathfrak{A}} := \{\varphi_0 \in \mathfrak{F}_0(X) |\ \exists K \in \mathfrak{A} : \varphi_0 \in \mathfrak{F}_0(K \cap U)\} |\ \mathfrak{A} \in \Phi_0, U \in \varphi \cap \tau\}$ is a base for a filter, which refines $\Phi_0^\uparrow$, and there must be an ultrafilter Φ_1, containing this base, therefore containing $\Phi_0^\uparrow$, too. Now, obviously $U \in \bigcap_{\varphi \in B_{U,\mathfrak{A}}} \varphi$ holds for every $U \in \varphi \cap \tau$, implying $\varphi \cap \tau \subseteq \Phi_1^{\curlyvee}$.
Otherwise, let $\Phi_0 \in \mathfrak{F}_0(\mathfrak{X}), A \in \mathfrak{X}$ and $(\Phi_0, A) \notin q_{\tau_l}$. This means, $\exists O \in \tau : A \cap O \neq \emptyset \wedge O^{-\mathfrak{x}} \notin \Phi_0$. Especially, there exists an element $a \in A \cap O$ and so O is an open neighbourhood of a. Now, Φ_0 is an ultrafilter on $\mathfrak{X}$, so $O^{-\mathfrak{x}} \notin \Phi_0$ just implies $O^{+\mathfrak{x}} \in \Phi_0$, leading to $\mathfrak{F}_0(O^c) \in \Phi_0^\uparrow$, yielding $\forall \Phi_1 \in \mathfrak{F}_0(\Phi_0^\uparrow) : \mathfrak{F}_0(O^c) \in \Phi_1$. But then we have $\forall \Phi_1 \in \mathfrak{F}_0(\Phi_0^\uparrow) : O^c \in \Phi_1^{\curlyvee}$ and therefore $O \notin \Phi_1^{\curlyvee}$. So, none of these $\Phi_1^{\curlyvee}$ contains $\overset{\bullet}{a} \cap \tau$. ■

125 Corollary
Let (X,τ) be a topological space, $\mathfrak{X} \subseteq \mathfrak{P}_0(X)$, $\Phi \in \mathfrak{F}_0(\mathfrak{X})$ and let A be a compact subset of X. Then Φ converges to A w.r.t. the Vietoris-topology, iff

(1) $\forall \Phi_1 \in \mathfrak{F}_0(\Phi^\uparrow) : \exists a \in A : (\Phi_1^{\curlyvee}, a) \in q_\tau$ *and*

(2) $\forall a \in A : \exists \Phi_1 \in \mathfrak{F}_0(\Phi_0^\uparrow) : (\Phi_1^{\curlyvee}, a) \in q_\tau$.

Proof: Let $\Phi \in \mathfrak{F}_0(\mathfrak{X})$ converge to A w.r.t. the Vietoris-topology. Because A is compact, every ultrafilter φ on A converges on A, i.e. it contains all open neighbourhoods of a point $a \in A$. But then $\Phi_1^{\curlyvee} \supseteq \varphi \cap \tau$ contains them, too. So, (1) follows from Proposition 123, and (2) follows from the fact, that $\overset{\bullet}{a}$ itself is an ultrafilter, together with proposition 124.
If otherwise Φ doesn't converge to A w.r.t. the Vietoris-topology, then it doesn't converge w.r.t. τ_l or w.r.t. τ_V^+. Then the second parts of the proofs of propositions 124 or 123, respectively, provide that (2) or (1), respectively, doesn't hold. ■

Now, we will go on to define convergences on the set of all filters on a topological space just by applying the requirements above to this case:

126 Definition
Let (X,τ) be a topological space and $\mathfrak{X} \subseteq \mathfrak{F}(X)$, then pseudotopological convergences $q_l(\tau)$ and $q_u(\tau)$ on $\mathfrak{X}$ are defined by

$$(\Psi,\psi) \in q_l(\tau) \quad :\Leftrightarrow \quad \forall \varphi \in \mathfrak{F}_0(\psi) : \exists \Phi_1 \in \mathfrak{F}_0(\Psi^\uparrow) : \Phi_1^{\bigcup\bigcap} \supseteq \varphi \cap \tau \ , \tag{5}$$
$$(\Psi,\psi) \in q_u(\tau) \quad :\Leftrightarrow \quad \forall \Phi_1 \in \mathfrak{F}_0(\Psi^\uparrow) : \exists \varphi \in \mathfrak{F}_0(\psi) : \Phi_1^{\bigcup\bigcap} \supseteq \varphi \cap \tau \tag{6}$$

for ultrafilters Ψ *on* $\mathfrak{X}$ *and filters* $\psi \in \mathfrak{X}$*, together with the "pseudotopological convention", that a filter on* $\mathfrak{X}$ *converges to an element of* $\mathfrak{X}$*, iff every refining ultrafilter does.*
A third convergence $q_V(\tau)$ is defined just by

$$(\Phi,\varphi) \in q_V(\tau) :\Leftrightarrow \forall \Phi_0 \in \mathfrak{F}_0(\Phi) : (\Phi_0,\varphi) \in q_l(\tau) \wedge (\Phi_0,\varphi) \in q_u(\tau) \ ,$$

and we call it the **strong Vietoris-pseudotopology** *on* $\mathfrak{F}(X)$*.*

In order to verify, that this defines indeed a pseudotopological convergence on $\mathfrak{F}(X)$, we have at first to remember, that our defining requirements only apply to ultrafilters and then the *generated* pseudotopology will be taken. So, it remains only to verify, that all singleton filters converge to their generating singleton - but this is very easy to see.
Although this convergence is quite strong, we will get a compactness result for this. For further investigations, our interest will focus a somewhat weaker, but quite similar convergence, defined here not for arbitrary filters, but for the compactoid ones.

127 Definition
Let (X,τ) be a topological space, then convergences $q_l'(\tau)$ and $q_u'(\tau)$ on $\mathfrak{C}(X)$ are defined by

$$(\Psi,\psi) \in q_l'(\tau) \quad :\Leftrightarrow \quad \forall \varphi \in \mathfrak{F}_0(\psi) : \exists \Phi_1 \in \mathfrak{F}_0(\Psi^\uparrow) : \forall A \in \psi : A \cap q_\tau(\Phi_1^{\bigcup\bigcap} \cap \varphi) \neq \emptyset \ ,$$
$$(\Psi,\psi) \in q_u'(\tau) \quad :\Leftrightarrow \quad \forall \Phi_1 \in \mathfrak{F}_0(\Psi^\uparrow) : \exists \varphi \in \mathfrak{F}_0(\psi) : \forall A \in \psi : A \cap q_\tau(\Phi_1^{\bigcup\bigcap} \cap \varphi) \neq \emptyset$$

for ultrafilters Ψ *on* $\mathfrak{F}(X)$*, together with the "pseudotopological convention", that a filter on* $\mathfrak{F}(X)$ *converges to a filter on* X*, iff every refining ultrafilter does.*
A third convergence $q_V'(\tau)$ is defined just by

$$(\Phi,\varphi) \in q_V'(\tau) :\Leftrightarrow \forall \Phi_0 \in \mathfrak{F}_0(\Phi) : (\Phi_0,\varphi) \in q_l'(\tau) \wedge (\Phi_0,\varphi) \in q_u'(\tau) \ ,$$

and we call it the **Vietoris-pseudotopology** *on* $\mathfrak{C}(X)$*.*

To check, that this really defines a pseudotopology is easy again by the same reasons as above.

128 Proposition
Let (X,τ) be a topological space. Then on the set $\mathfrak{C}(X)$ of all compactoid filters on X hold

$$\begin{aligned} q_l(\tau) &\subseteq q_l'(\tau)\ , \\ q_u(\tau) &\subseteq q_u'(\tau)\ , \textit{ and consequently} \\ q_V(\tau) &\subseteq q_V'(\tau)\ . \end{aligned}$$

Proof: If Ψ,ψ fulfill the requirements to converge in one of the senses of definition 126, the corresponding requirement of definition 127 is fulfilled with the same Φ_1 respectively φ_0. We have just to observe, that a filter, which contains all open members of a convergent filter, converges at least to the same points. ■

From this and from corollary 125 we see, that $q_V'(\tau)$ coincides with the Vietoris-topology on $K(X)$, provided, we identify the compact sets with their generated principal filters.

129 Lemma
Let (X,τ) be a compact topological space. Then $(\mathfrak{C}(X), q_V(\tau))$ is compact, too.

Proof: Let $\Phi \in \mathfrak{F}_0(\mathfrak{C}(X))$. We will show, that Φ converges in $q_V(\tau)$ to the filter

$$\varphi_\Phi := \left\langle \left\{ adh(\bigcap_{\chi\in\mathfrak{A}} \chi) \;\middle|\; \mathfrak{A}\in\Phi \right\} \right\rangle\ ,$$

which is compactly generated, and thus compactoid, because (X,τ) is compact, and so the (by proposition 36 closed) generating sets are compact, too.
To prove $(\Phi,\varphi_\Phi) \in q_l(\tau)$, let $\varphi_0 \in \mathfrak{F}_0(\varphi_\Phi)$ be given. Then we have for all $U\in\varphi_0\cap\tau$, that $\forall\mathfrak{A}\in\Phi : U\cap\overline{\bigcup_{\chi\in\mathfrak{A}} adh(\chi)} \neq \emptyset$ and therefore $U\cap\bigcup_{\chi\in\mathfrak{A}} adh(\chi)\neq\emptyset$, because of the closedness-properties and the fact, that U is open. Thus, for all $U\in\varphi_0\cap\tau$ and all $\mathfrak{A}\in\Phi$, the set

$$M_{U,\mathfrak{A}} := \{\psi\in\mathfrak{F}_0(X)|\ \exists\chi\in\mathfrak{A}, u\in U : \psi\supseteq\chi\wedge(\psi,u)\in q_\tau\}$$

is not empty. Obviously, for $U_1,U_2\in\varphi_0\cap\tau$ and $\mathfrak{A}_1,\mathfrak{A}_2\in\Phi$ we get $M_{U_1\cap U_2,\mathfrak{A}_1\cap\mathfrak{A}_2}\subseteq M_{U_1,\mathfrak{A}_1}\cap M_{U_2\cap\mathfrak{A}_2}$, so $\mathfrak{M} := \{M_{U,\mathfrak{A}}|\ U\in\varphi_0\cap\tau,\mathfrak{A}\in\Phi\}$ is a filterbase, and there exists an ultrafilter Φ_1, which contains $\mathfrak{M}$. Observe now, that $M_{U,\mathfrak{A}}\in\Phi_1$ for all $\mathfrak{A}\in\Phi, U\in\varphi_0\cap\tau$ and $M_{U,\mathfrak{A}}\subseteq\bigcup_{\chi\in\mathfrak{A}}\mathfrak{F}_0(\chi)\subseteq\mathfrak{F}_0(\bigcap_{\chi\in\mathfrak{A}}\chi)$ together imply $\Phi_1\supseteq\Phi^\uparrow$. Furthermore, every $\psi\in M_{U,\mathfrak{A}}$ converges to an element of U, so it must contain the open set U, yielding $U\in\bigcup_{\psi\in M_{U,\mathfrak{A}}}\psi$. Now, all $M_{U,\mathfrak{A}}$ with $U\in\varphi\cap\tau$ and $\mathfrak{A}\in\Phi$ belong to Φ_1, which implies $\Phi_1^{\bigcup\bigcap}\supseteq\varphi_0\cap\tau$. So, the defining requirement for $q_l(\tau)$ in 126(5) is fulfilled.
To prove, $(\Phi,\varphi_\Phi)\in q_u(\tau)$, let $\Phi_1\in\mathfrak{F}_0(\Phi^\uparrow)$ be given. Because (X,τ) is compact, every ultrafilter on X converges w.r.t. q_τ, i.e. $\forall\psi\in\mathfrak{F}_0(X) : q_\tau(\psi)\neq\emptyset$. So, there exists a map $\lambda:\mathfrak{F}_0(X)\to X$ with $\forall\psi\in\mathfrak{F}_0(X) : \lambda(\psi)\in q_\tau(\psi)$. Now, $\varphi_0 := \lambda(\Phi_1)$ is an

ultrafilter on X, because Φ_1 is an ultrafilter on $\mathfrak{F}_0(X)$. Moreover, $\varphi_0 \supseteq \varphi_\Phi$ holds, because $\forall T \in \varphi_\Phi : \exists \mathfrak{A} \in \Phi : T \supseteq adh(\bigcap_{\chi\in\mathfrak{A}} \chi) = q_\tau(\mathfrak{F}_0(\bigcap_{chi\in\mathfrak{A}} \chi)) \supseteq \lambda(\mathfrak{F}_0(\bigcap_{\chi\in\mathfrak{A}} \chi)) \in \lambda(\Phi^\uparrow) \subseteq \lambda(\Phi_1)$. Now, let $U \in \varphi_0 \cap \tau$. Then there is an $M \in \Phi_1$, s.t. $\lambda(M) \subseteq U$, i.e. all elements of M converge to elements of U, so they all must contain the open neighbourhood U. But then $U \in \bigcup_{\psi\in M} \psi$ holds. This is valid for all $U \in \varphi_0 \cap \tau$, yielding $\varphi_0 \cap \tau \subseteq \Phi_1^{\bigcup\bigcap}$; so the defining requirement for $q_u(\tau)$ in 126(6) is fulfilled, too.■

130 Corollary
Let (X, τ) be a compact topological space. Then $(\mathfrak{C}(X), q'_V(\tau))$ is compact, too.

Proof: Follows directly from lemma 129 and proposition 128. ■

4.2 A Hyperstructure for Limited Multifilter - Spaces

If $A_1, ..., A_n$ are subsets of a set X and $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$, then let

$$< A_1, ..., A_n >_{\mathfrak{A}} := \{M \in \mathfrak{A} |\ M \subseteq \bigcup_{i=1}^{n} A_i \wedge \forall i = 1, ..., n : M \cap A_i \neq \emptyset\} .$$

Now, for $\alpha \subseteq \mathfrak{P}_0(X)$ we set

$$\alpha_{V,\mathfrak{A}} := \{< A_1, ..., A_n > |\ n \in I\!N, A_i \in \alpha\}$$

and for $\Sigma \in \widehat{\mathfrak{F}}(X)$ we define

$$\Sigma_{V,\mathfrak{A}} := [\{\alpha_{V,\mathfrak{A}} |\ \alpha \in \Sigma\}]_{\widehat{\mathfrak{F}}(\mathfrak{A})} .$$

For brevity, we will simply write $< A_1, ..., A_n >, \alpha_V$ and Σ_V, if it is clear from the context, which $\mathfrak{A}$ is regarded.

Note, that $\{\alpha_V |\ \alpha \in \Sigma\}$ is indeed a base for Σ_V, because from $\alpha \preceq \beta$ always follows $\alpha_V \preceq \beta_V$ (for $< A_1, ..., A_n > \in \alpha_V$, there are $B_i \in \beta$ s.t. $A_i \subseteq B_i, i = 1, ..., n$, simply implying $< A_1, ..., A_n > \subseteq < B_1, ..., B_n > \in \beta_V$).

131 Definition
Let $(X, \mathcal{M})$ be a limited multifilter-space. Then we call

$$\mathcal{M}_V := \{\underline{\Sigma} \in \widehat{\mathfrak{F}}(\mathcal{PC}(X)) |\ \exists \Xi \in \mathcal{M} : \underline{\Sigma} \preceq \Xi_{V,\mathcal{PC}(X)}\}$$

the **finite hyperstructure** *on $\mathcal{PC}(X)$ w.r.t. $\mathcal{M}$.*

132 Proposition
If $(X, \mathcal{M})$ is a limited multifilter-space, then $(\mathcal{PC}(X), \mathcal{M}_V)$ is a limited multifilter-space, too.

Proof: To show, that $(\mathcal{PC}(X), \mathcal{M}_V)$ is indeed a multifilter-space, we have only to verify, that all singleton-multifilters on $\mathcal{PC}(X)$ belong to $\mathcal{M}_V$, because the closedness of $\mathcal{M}_V$ against refinement of multifilters is ensured by definition. For each precompact subset P of X we have a $\Sigma \in \mathcal{M}$ s.t. $\forall \alpha \in \Sigma : \exists n \in I\!N, A_1, ..., A_n \in \alpha : P \subseteq \bigcup_{i=1}^{n} A_i$, implying $P \in < A_{j_1}, ..., A_{j_m} > \in \alpha_V$ for $\{A_{j_1}, ..., A_{j_m}\} := \{A_i | \ 1 \leq i \leq n, P \cap A_i \neq \emptyset\}$, implying $\{\{P\}\} \preceq \alpha_V$, thus $\widehat{P} \preceq \Sigma_V \in \mathcal{M}_V$.
Let $\underline{\Sigma_1}, \underline{\Sigma_2} \in \mathcal{M}_V$, then there are $\Xi_1, \Xi_2 \in \mathcal{M}$ with $\underline{\Sigma_i} \preceq \Xi_{iV}, i = 1, 2$. Now, $\underline{\Sigma_1} \cap \underline{\Sigma_2} \preceq (\Xi_1 \cap \Xi_2)_V \in \mathcal{M}_V$ follows immediately from the fact, that each union of the families of finite subsets of members $\xi_1 \in \Xi_1$ and $\xi_2 \in \Xi_2$, respectively, is a subset of the family of finite subsets of $\xi_1 \cup \xi_2$. ■

133 Theorem
Let $(X, \mathcal{M})$ be a limited multifilter-space. Then $(\mathcal{PC}(X), \mathcal{M}_V)$ is precompact, if and only if $(X, \mathcal{M})$ is precompact.

Proof: Let $(X, \mathcal{M})$ be precompact. By corollary 74, there exists $\Sigma \in \mathcal{M}$, s.t. $\forall \alpha \in \Sigma : \exists n_\alpha \in I\!N, A_1(\alpha), ..., A_{n_\alpha}(\alpha) \in \alpha : \bigcup_{i=1}^{n_\alpha} A_i(\alpha) = X$. Then $\Sigma_V \in \mathcal{M}_V$ holds and for every $\alpha_V \in \Sigma_V$ we have for each (necessarily precompact) subset P of X, that $P \in < A_{j_1}(\alpha), ..., A_{j_m}(\alpha) > \in \alpha_V$ holds for $\{A_{j_1}(\alpha), ..., A_{j_m}(\alpha)\} := \{A_i(\alpha) | \ 1 \leq i \leq n_\alpha, P \cap A_i(\alpha) \neq \emptyset\}$. So, the families $< A_{j_1}(\alpha), ..., A_{j_m}(\alpha) >$, taken for all subsets $\{A_{j_1}(\alpha), ..., A_{j_m}(\alpha)\}$ of $\{A_1(\alpha), ..., A_{n_\alpha}(\alpha)\}$, cover $\mathfrak{P}_0(X)$ completely. But $\{A_1(\alpha), ..., A_{n_\alpha}(\alpha)\}$ has only finitely many subsets.
If otherwise $(\mathcal{PC}(X), \mathcal{M}_V)$ is precompact, from proposition 132 and corollary 74 follows the existence of an $\Sigma \in \mathcal{M}$, s.t. $\forall \alpha \in \Sigma : \exists m, n_1, ..., n_m \in I\!N, A_i^{(j)} \in \alpha, 1 \leq j \leq m, 1 \leq i \leq n_j : \mathcal{PC}(X) \subseteq \bigcup_{j=1}^{m} < A_1^{(j)}, ..., A_{n_j}^{(j)} >$. Now, all singletons $\{x\}, x \in X$ are precompact and consequently each $x \in X$ is contained in some $\bigcup_{i=1}^{n_j} A_i^{(j)}$, yielding $X \subseteq \bigcup_{j=1}^{m} (\bigcup_{i=1}^{n_j} A_i^{(j)})$. ■

134 Lemma
If $(X, \mathcal{M})$ is a limited multifilter-space and $\mathfrak{A} \subseteq \mathcal{PC}(X)$, then $\mathfrak{A}$ is precompact w.r.t. $\mathcal{M}_V$ if and only if $\bigcup_{A \in \mathfrak{A}} A$ is precompact w.r.t. $\mathcal{M}$.

Proof: If $\bigcup_{A \in \mathfrak{A}} A$ is precompact, then $\mathcal{PC}(\bigcup_{A \in \mathfrak{A}} A)$ is precompact by theorem 133, thus its subset $\mathfrak{A}$ is (because every precompact subset of X clearly is precompact in $\bigcup_{A \in \mathfrak{A}} A$). So, let $\mathfrak{A}$ be precompact w.r.t. $\mathcal{M}_V$. Now, $(\mathcal{PC}(X), \mathcal{M}_V)$ is limited by proposition 132, so by corollary 74 there must exist a $\Sigma \in \mathcal{M}$ with $\forall \sigma \in \Sigma : \exists m, n_1, ..., n_m \in I\!N, S_1^{(1)}, ..., S_{n_m}^{(m)} \in \sigma : \mathfrak{A} \subseteq \bigcup_{j=1}^{m} < S_1^{(j)}, ..., S_{n_j}^{(j)} >$, implying $\forall A \in \mathfrak{A} : \exists j \in \{1, ..., m\} : A \subseteq \bigcup_{i=1}^{n_j} S_i^{(j)}$ and consequently $\bigcup_{A \in \mathfrak{A}} A \subseteq \bigcup_{j=1}^{m} \bigcup_{i=1}^{n_j} S_i^{(j)}$, yielding $\bigcup_{A \in \mathfrak{A}} A$ being precompact w.r.t. $\mathcal{M}$ by corollary 74. ■

5 Mizokami-maps and Ascoli-Theorems

5.1 Topological Base Spaces

In the famous paper [25], Mizokami proved, that for Hausdorff topological spaces $(X,\tau),(Y,\sigma)$ the function space $C(X,Y)$, endowed with the compact-open topology, can be embedded as a closed subspace of the function space $C(K(X),K(Y))$, endowed with the pointwise convergence, where the hyperspaces are equipped with Vietoris topology. Just the same kind of map was used by Edwards in his paper [15], to get a very nice looking and surprising Ascoli-theorem (3.13 in [15]) for the compact-open topology – without any requirement on the set of functions in question to be evenly continuous or similar, and with quite weak assumtions about the range space. Unfortunately, his statement is not true, as we will see.

Here we will use the mentioned kind of mapping to prove Ascoli-like theorems for set-open topologies with only weak assumtions about the range space, too.

For the Vietoris-topology on any $\mathfrak{B}\subseteq\mathfrak{P}_0(Z)$ for a topological space Z we will use the base consisting of all sets

$$<O_1,...,O_n>_{\mathfrak{B}}:=\mathfrak{B}\cap\{M\in\mathfrak{P}_0(Z)|\ n\in I\!N, M\subseteq\bigcup_{i=1}^{n}O_i,\ \forall i: M\cap O_i\neq\emptyset\}$$

with open subsets O_i. If there seems to be no doubt, the index $\mathfrak{B}$ will be omitted from $<O_1,...,O_n>$. Let $(X,\tau),(Y,\sigma)$ be topological spaces and $\mathfrak{A}\subseteq\mathfrak{P}(X)$. By $C_Y(\mathfrak{A})$ we denote the set $C_Y(\mathfrak{A}):=\{f(A)|\ A\in\mathfrak{A}, f\in C(X,Y)\}$ of all continuous images in Y of members of $\mathfrak{A}$. Now, we can naturally map the set Y^X, into the set $\mathfrak{P}(Y)^{\mathfrak{A}}$:

$$\mu: Y^X\to\mathfrak{P}(Y)^{\mathfrak{A}}: f\to\mu(f): A\to f(A)$$

135 Proposition
Let $(X,\tau),(Y,\sigma)$ be topological spaces and $\mathfrak{A}\subseteq\mathfrak{P}(X)$. If the function $f:X\to Y$ is continuous, then the function $\mu(f):\mathfrak{A}\to\mathfrak{P}(Y)$ is continuous w.r.t. the Vietoris-topologies on $\mathfrak{A}$ and $\mathfrak{P}(Y)$.
If $\mu(f)$ is continuous and $\mathfrak{A}$ is closed under finite unions and has the properties

(1) $\forall V\in\sigma, x\in f^{-1}(V):\exists A_x\in\mathfrak{A}: x\in A_x\subseteq f^{-1}(V)$ *and*

(2) $\forall O\in\tau:\exists\mathfrak{B}\subseteq\mathfrak{A}: \bigcup_{B\in\mathfrak{B}}B=O$,

then f is continuous, too.

Proof: Let $<V_1,...,V_n>$ be an open base set of σ_V with all $V_i\in\sigma$. Then we have $A\in\mu(f)^{-1}(<V_1,...,V_n>)\Leftrightarrow A\in\mathfrak{A}\wedge f(A)\in<V_1,...,V_n>\Leftrightarrow A\in$

$\mathfrak{A} \wedge f(A) \subseteq \bigcup_{i=1}^n V_i \wedge \forall i : f(A) \cap V_i \neq \emptyset \Leftrightarrow A \in \mathfrak{A} \wedge A \subseteq \bigcup_{i=1}^n f^{-1}(V_i) \wedge \forall i : A \cap f^{-1}(V_i) \neq \emptyset \Leftrightarrow A \in < f^{-1}(V_1), ..., f^{-1}(V_n) >_{\mathfrak{A}}$. Thus $\mu(f)^{-1}(< V_1, ..., V_n >) =< f^{-1}(V_1), ..., f^{-1}(V_n) >$ is an open base set of τ_V on $\mathfrak{A}$, because all $f^{-1}(V_i)$ are open by the continuity of f.
Let $\mathfrak{A}$ have the mentioned properties, $\mu(f)$ be continuous and $V \in \sigma$. Then $(\mu(f))^{-1}(< V >)$ is open in τ_V, i.e. $\forall A \in (\mu(f))^{-1}(< V >) : \exists U_1(A), ..., U_{k(A)}(A) \in \tau : A \in < U_1(A), ..., U_{k(A)}(A) > \subseteq (\mu(f))^{-1}(< V >)$. Now, by (1) we find $\forall x \in f^{-1}(V) : \exists A_x \in \mathfrak{A} : x \in A_x \subseteq f^{-1}(V)$, implying $A_x \in \mu(f)^{-1}(< V >)$. So, there are $U_1(A_x), ..., U_{k(A_x)}(A_x) \in \tau$ s.t. $A_x \in < U_1(A_x), ..., U_{k(A_x)}(A_x) >_{\mathfrak{A}} \subseteq \mu(f)^{-1}(< V >)$, so by property (2) we get $\forall i = 1, ..., k(A_x) : \exists \mathfrak{B}_i \subseteq \mathfrak{A} : \bigcup_{B \in \mathfrak{B}_i} B = U_i(A_x)$ and then we take $\mathfrak{C} := \{\bigcup_{i=1}^{k(A_x)} B_i |\ \forall i : B_i \in \mathfrak{B}_i\}$ which is a subset of $\mathfrak{A}$ by closedness under finite unions. Now, we have $\bigcup_{C \in \mathfrak{C}} C = \bigcup_{i=1}^{k(A_x)} U_i(A_x)$, so obviously $\mathfrak{C} \subseteq < U_1(A_x), ..., U_{k(A_x)}(A_x) >$, which is contained in $(\mu(f))^{-1}(< V >)$, implying $\forall C \in \mathfrak{C} : \mu(f)(C) \subseteq V$ and therefore $f(\bigcup_{C \in \mathfrak{C}} C) = \bigcup_{C \in \mathfrak{C}} \mu(f)(C) \subseteq V$, implying $\bigcup_{C \in \mathfrak{C}} C \subseteq f^{-1}(V)$, so $O_x := \bigcup_{C \in \mathfrak{C}} C$ $(= \bigcup_{i=1}^{k(A_x)} U_i(A_x))$ is an open neighbourhood of x, contained in $f^{-1}(V)$. Taking these O_x for all $x \in f^{-1}(V)$ we find $f^{-1}(V)$ to be open. ■

If $\mathfrak{A}$ contains the finite subsets of X, then it has obviously all the properties required in the second part of the proposition. In any case, proposition 135 ensures, that the image of $C(X, Y)$ under the mapping μ is a subset of $C(\mathfrak{A}, C_Y(\mathfrak{A}))$, where $\mathfrak{A}$ and $C_Y(\mathfrak{A})$ are equipped with Vietoris topology.

136 Proposition
Let $(X, \tau), (Y, \sigma)$ be topological spaces, $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ and $\mathcal{H} \subseteq Y^X$. Then the map

$$\mu : \mathcal{H} \to \mu(\mathcal{H}) := \{\mu(f) |\ \mu(f) : A \to f(A), f \in \mathcal{H}\} \subseteq \mathfrak{P}_0(Y)^{\mathfrak{A}}$$

is open, where $\mathfrak{A}$ and $\mathfrak{P}_0(Y)$ are equipped with Vietoris topology and $\mathfrak{P}_0(Y)^{\mathfrak{A}}$ with pointwise topology.
If $\mathcal{H} \subseteq C(X, Y)$ and $\mathfrak{A}$ has the property

$$\forall O \in \tau, A \in \mathfrak{A} : O \cap A \neq \emptyset \Rightarrow \exists A_O \in \mathfrak{A} : A_O \subseteq A \cap O\ , \tag{7}$$

then this map is continuous.

Proof: Let $\mathfrak{O} := \bigcap_{i=1}^n (A_i, O_i)$ with $A_i \in \mathfrak{A}, O_i \in \sigma$ be a basic open set of $\tau_{\mathfrak{A}}$. Then holds $f \in \mathfrak{O} \Leftrightarrow \forall i \in \{1, ..., n\} : f(A_i) \subseteq O_i \Leftrightarrow \forall i \in \{1, ..., n\} : \mu(f)(A_i) \in < O_i > \Leftrightarrow \mu(f) \in \bigcap_{i=1}^n (\{A_i\}, < O_i >)$, yielding $\mu(\mathfrak{O}) = \bigcap_{i=1}^n (\{A_i\}, < O_i >)$, which is an basic open set of the pointwise topology on $\mu(\mathcal{H})$.
Let $(\mathcal{F}, f) \in q_{\tau_{\mathfrak{A}}}$, so by taking principal filters in proposition 87, we get

$$\forall A \in \mathfrak{A} : \mathcal{F}(A) \supseteq [f(A)] \cap \sigma\ . \tag{8}$$

Now, let $A_0 \in \mathfrak{A}$ be given with $f(A_0) \in < V_1, ..., V_n >$ for some $V_1, ..., V_n \in \sigma$. This means $f(A_0) \subseteq V_0 := \bigcup_{i=1}^n V_i$ and $\forall i \in \{1, ..., n\} : f(A_0) \cap V_i \neq \emptyset$, implying $\forall i \in \{1, ..., n\} : \exists A_i \in \mathfrak{A} : A_i \subseteq A_0 \cap f^{-1}(V_i)$, because of the required property of $\mathfrak{A}$ and the continuity of f. Then from (8) follows $\forall j \in \{0, 1, ..., n\} : \exists F_j \in \mathcal{F} : F_j(A_j) \subseteq V_j$, just meaning $\forall g \in F_j : g(A_j) \subseteq V_j$, thus from $A_j \subseteq A_0$ we get $\forall g \in F_j : g(A_0) \cap V_j \neq \emptyset$ and especially for $j = 0$ we have $F_0(A_0) \subseteq V_0$. But then $F := \bigcap_{j=0}^n F_j$ is an element of $\mathcal{F}$ and fulfills $\mu(F)(A_0) \subseteq < V_1, ..., V_n >$. This is valid for all basic open neighbourhoods of $f(A_0)$, so $\mu(\mathcal{F})(A_0)$ converges to $f(A_0) = \mu(f)(A_0)$ w.r.t. σ_V – for all $A_0 \in \mathfrak{A}$, thus $\mu(\mathcal{F})$ converges pointwise to $\mu(f)$. ■

The property (7) is trivially fulfilled, if $\mathfrak{A}$ contains the singletons. Moreover, in this case we don't need to restrict the map to $C(X, Y)$, in order to prove its continuity.

137 Lemma
Let $(X, \tau), (Y, \sigma)$ be topological spaces, let $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ contain the singletons and $\mathcal{H} \subseteq Y^X$. Then the map

$$\mu : \mathcal{H} \to \mu(\mathcal{H}) := \{\mu(f) |\ \mu(f) : A \to f(A), f \in \mathcal{H}\} \subseteq \mathfrak{P}_0(Y)^{\mathfrak{A}}$$

is open, continuous and bijective, where $\mathcal{H}$ is equipped with the $\mathfrak{A}$-open topology and $\mathfrak{P}_0(Y)^{\mathfrak{A}}$ with the pointwise from the Vietoris topology on $\mathfrak{P}_0(Y)$.

Proof: It's easy to see, that it is bijective, because each function f from X to Y is uniquely determined by the images of $\mu(f)$ on the singletons. Proposition 136 says, that it is open and, as is easy to see, the proof of continuity in proposition 136 will work fine even without continuity of the $\tau_{\mathfrak{A}}$-limit function f of the filter $\mathcal{F}$, if we have in $\mathfrak{A}$ all singletons, because the combination of property (7) and continuity of f is only needed to ensure the existence of the subsets $\mathfrak{A} \ni A_i \subseteq A_0 \cap f^{-1}(V_i)$ for $i = 1, ..., n$, but now we can always take singletons $\{x_i\}$ instead of these A_i. ■

We will call this map

$$\mu : (Y^X, \tau_{\mathfrak{A}}) \to (\mu(Y^X), \tau_p) \subseteq (\mathfrak{P}_0(Y)^{\mathfrak{A}}, \tau_p) : f \to \mu(f) : A \to f(A)$$

the *Mizokami–map*, where $\mathfrak{A}$ and $\mathfrak{P}_0(Y)$ are endowed with Vietoris topology.

138 Proposition
Let $(X, \tau), (Y, \sigma)$ be topological spaces and let $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ contain the singletons. If $\mathcal{F}$ is a filter on Y^X s.t. $\mu(\mathcal{F}) \xrightarrow{p} g \in \mathfrak{P}_0(Y)^{\mathfrak{A}}$, where $\mathfrak{P}_0(Y)$ is equipped with Vietoris topology, then there exists $g' \in Y^X$, with $\forall x \in X : g'(x) \in g(\{x\})$ and $\mathcal{F} \xrightarrow{p} g'$.

Proof: $\mu(\mathcal{F}) \xrightarrow{p} g$ yields for each singleton $\{x\} \subseteq X$, that $g(\{x\}) \in < V, Y >$ with $V \in \sigma$ implies $\exists F \in \mathcal{F} : \forall f \in F : f(x) \in V$. Now, $g(\{x\})$ is never the empty set $\emptyset$, because this is not an element of our range space, so there exists a function

$g' : X \to Y$ with $g'(x) \in g(\{x\})$ for all $x \in X$. But for arbitrary $y_x \in g(\{x\})$ and $V \in \overset{\bullet}{y_x} \cap \sigma$ we find $g(\{x\}) \in < V, Y >$, and consequently $V \in \mathcal{F}(x)$. Thus $\mathcal{F}(x) \overset{\sigma}{\to} y_x$ and therefore $\mathcal{F}$ converges pointwise to g'. ■

139 Definition
Let $(X,\tau),(Y,\sigma)$ be topological spaces and $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$. A subset $\mathcal{H} \subseteq Y^X$ is said to be **$\mathfrak{A}$-evenly continuous**, *iff for all $A \in \mathfrak{A}$ holds*

$$\forall \mathcal{F} \in \mathfrak{F}_0(\mathcal{H}), \varphi \in \mathfrak{F}(A), x \in X : (\mathcal{F}(x) \overset{\sigma}{\to} y) \wedge (\varphi \overset{\tau}{\to} x) \;\Rightarrow\; \mathcal{F}(\varphi) \overset{\sigma}{\to} y \ .$$

$\mathcal{H}$ is said to be **evenly continuous**, *iff it is $\{X\}$-evenly continuous.*
$\mathcal{H}$ is said to be **evenly continuous on a subset** *K, iff the set of restricted functions $\mathcal{H}_{|K} := \{f_{|K} : K \to Y |\ f \in \mathcal{H}\}$ is evenly continuous.*

140 Proposition
Let $(X,\tau),(Y,\sigma)$ be topological spaces and $\mathcal{H} \subseteq C(X,Y)$.
If $\mathcal{H}$ is $\{K\}$-evenly continuous for a subset $K \subseteq X$, then it is evenly continuous on K.
If Y Hausdorff, K a compact subset of X, and $\mathcal{H}$ evenly continuous on K, then it is $\{K\}$-evenly continuous.

Proof: The first statement follows trivially from the definition. So, let Y be Hausdorff, K compact and $\mathcal{H}$ be evenly continuous on K.
Furthermore, let $\mathcal{F}$ be a filter on $\mathcal{H}$, $x \in X$, $\varphi \in \mathfrak{F}(K)$ s.t. $\varphi \to x$ and $\mathcal{F}(x) \to y \in Y$. Now, we have for each refining ultrafilter φ_0 of φ, that it converges to x, too. But it must also converge to an element $a \in K$. Then for all continuous functions f follows $f(\varphi_0) \to f(a)$ and $f(\varphi_0) \to f(x)$, yielding $f(a) = f(x)$, because of the Hausdorffness of Y. Thus $\mathcal{F}(a) = \mathcal{F}(x)$, because all members of $\mathcal{F}$ consist of continuous functions. Consequently, $\mathcal{F}(a) \to y$, thus $\mathcal{F}(\varphi_0) \to y$, too, because $\mathcal{H}$ is evenly continuous on K. So, for an arbitrary $V \in \overset{\bullet}{y} \cap \sigma$ there must exist $F \in \mathcal{F}, P \in \varphi_0$, s.t. $F(P) \subseteq V$. Obviously, the family $\mathfrak{A}_V := \{A \subseteq X |\ \exists F \in \mathcal{F} : F(A) \subseteq V\}$ is closed under finite unions, because $\mathcal{F}$ is closed under finite intersections, and we have seen, that $\varphi_0 \cap \mathfrak{A}_V \neq \emptyset$ for every refining ultrafilter φ_0 of φ. So, lemma 9 applies, yielding $\varphi \cap \mathfrak{A}_V \neq \emptyset$. This is valid for all open neighbourhoods of y, implying $\mathcal{F}(\varphi) \to y$. ■

141 Proposition
Let $(X,\tau),(Y,\sigma)$ be topological spaces, Y Hausdorff, and let $\mathcal{H}$ be a relative compact subset of $C(X,Y)$ w.r.t. the compact-open topology τ_{co}. Then $\mathcal{H}$ is evenly continuous on all compact subsets of X.

Proof: Let $A \subseteq X$ be compact, $\varphi \in \mathfrak{F}(A)$, $a \in A$ and $\mathcal{F} \in \mathfrak{F}(\mathcal{H})$, s.t. $\mathcal{F}(a) \to y \in Y$ and $\varphi \to a$.
Then each refining ultrafilter $\mathcal{F}_0$ of $\mathcal{F}$ τ_{co}-converges to a continuous function g, because of the relative compactness of $\mathcal{H}$ in $C(X,Y)$. So, $y = g(a)$ follows, because

$\mathcal{F}_0(a) \to y$, $\mathcal{F}_0$ converges especially pointwise to g and Y is Hausdorff. Moreover, $g(\varphi) \to y = g(a) \in g(A)$ holds, and $g(A)$ is compact and therefore closed, because A is compact, thus $g(A)$ is T_3, because Y is Hausdorff. Now, let $V_0 \in \overset{\bullet}{y} \cap \sigma$, then there exists $V_1 \in \sigma$, s.t. $y \in V_1 \cap g(A) \subseteq \overline{V_1 \cap g(A)} \subseteq V_0 \cap g(A)$. Furthermore, there exists $P_1 \in \varphi$, s.t. $g(P_1) \subseteq V_1 \cap g(A) \subseteq \overline{V_1 \cap g(A)}$ (remember, φ is a filter on A) and consequently $g^{-1}(\overline{V_1 \cap g(A)}) \in \varphi$ and $g^{-1}(\overline{V_1 \cap g(A)})$ is closed in X, thus $B := g^{-1}(\overline{V_1 \cap g(A)}) \cap A$ is compact. But $g(B) \subseteq \overline{V_1 \cap g(A)} \subseteq V_0$ holds and $\mathcal{F}_0$ converges w.r.t. τ_{co} to g, thus $(B, V_0) \in \mathcal{F}_0$ and we have $B \in \varphi$, so $V_0 \in \mathcal{F}_0(\varphi)$ follows. Now, the family $\mathfrak{A}_{V_0} := \{F \subseteq \mathcal{H} |\ \exists P \in \varphi : F(P) \subseteq V_0\}$ is closed under finite unions of it's members, because φ is closed under finite intersections, and we have seen, that every refining ultrafilter of $\mathcal{F}$ contains a member of $\mathfrak{A}_{V_0}$. Thus, lemma 9 applies, yielding $\mathcal{F} \cap \mathfrak{A}_{V_0} \neq \emptyset$, and this is valid for every $V_0 \in \overset{\bullet}{y} \cap \sigma$. So, $\mathcal{F}(\varphi)$ converges to y.■

142 Lemma
Let $(X, \tau), (Y, \sigma)$ be topological spaces, $R \subseteq X$ a compact (resp. relative compact) subset and let $\mathcal{H} \subseteq C(X, Y)$ be $\{R\}$-evenly continuous. Then holds:
If for every ultrafilter on R among its convergence-points exists a point $x \in R$ (resp. $x \in X$), s.t. the set $\mathcal{H}(x) := \{f(x) |\ f \in \mathcal{H}\}$ is compact (resp. relative compact) in Y, then $\mathcal{H}(R) := \{f(x) |\ f \in \mathcal{H}, x \in R\}$ is compact (relative compact) in Y, too.

Proof: Let $\psi \in \mathfrak{F}_0(\mathcal{H}(R))$. We have $\forall y \in \mathcal{H}(R) : \exists x_y \in R, f_y \in \mathcal{H} : y = f_y(x_y)$, thus there exists a map $\pi : \mathcal{H}(R) \to R \times \mathcal{H} : \pi(y) = (x_y, f_y), f_y(x_y) = y$. Now, $\pi(\psi)$ is an ultrafilter on $R \times \mathcal{H}$ and consequently $\pi_1(\pi(\psi))$ and $\pi_2(\pi(\psi))$ are ultrafilters on R and $\mathcal{H}$, respectively, where $\pi_1 : R \times \mathcal{H} \to R$ and $\pi_2 : R \times \mathcal{H} \to \mathcal{H}$ are the canonical projections. So, $\pi_1(\pi(\psi))$ converges to a point $x_0 \in R$ (resp. $x_0 \in X$), s.t. $\mathcal{H}(x_0)$ is compact (resp. relative compact) in Y. Furthermore, $\pi_2(\pi(\psi))(x_0)$ is an ultrafilter on $\mathcal{H}(x_0)$, thus it converges in $\mathcal{H}(x_0) \subseteq \mathcal{H}(R)$ (resp. in Y) to a point y_0. But then the $\{R\}$-even continuity of $\mathcal{H}$ implies that $\pi_2(\pi(\psi))(\pi_1(\pi(\psi)))$ converges to y_0, too. But we have naturally $\pi_2(\pi(\psi))(\pi_1(\pi(\psi))) \subseteq \psi$, so ψ converges in $\mathcal{H}(R)$ (resp. in Y). ■

143 Lemma
(Essential Ascoli)
Let $(X, \tau), (Y, \sigma)$ be topological spaces and $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$. Let $\mathcal{H} \subseteq C(X, Y)$ and $\mathcal{F}$ be an ultrafilter on $\mathcal{H}$, which converges pointwise to a function $g \in C(X, Y)$. Then hold:

(1) If $\mathfrak{A}$ consists only of relative compact subsets of X, $\mathcal{H}$ is $\mathfrak{A}$-evenly continuous and the images of all members of $\mathfrak{A}$ under g are closed in Y, then $\mu(\mathcal{F})$ converges pointwise to $\mu(g)$ in $C(\mathfrak{A}, C_Y(\mathfrak{A}))$.

(2) If $\mathfrak{A}$ consists only of compact subsets of X and $\mathcal{H}$ is evenly continuous on all members of $\mathfrak{A}$, then $\mu(\mathcal{F})$ converges pointwise to $\mu(g)$ in $C(\mathfrak{A}, C_Y(\mathfrak{A}))$.

Proof: The continuity of $\mu(g)$ is ensured by proposition 135.
Assume, $\mu(\mathcal{F})$ would not converge pointwise to $\mu(g)$. Then there are $A \in \mathfrak{A}$ and $V_1, ..., V_n \in \sigma$ such that $g(A) \in < V_1, ..., V_n >$, but $\forall F \in \mathcal{F} : \exists f \in F : f(A) \notin < V_1, ..., V_n >$. Thus, $\{f \in \mathcal{H} |\ f(A) \notin \bigcup_{i=1}^n V_i\} \cup \bigcup_{i=1}^n \{f \in \mathcal{H} |\ f(A) \cap V_i = \emptyset\}$ is a member of $\mathcal{F}$, because it's complement is not. Because $\mathcal{F}$ is an ultrafilter, one of the unified sets above must itself belong to $\mathcal{F}$, by proposition 7.
Assume, it would hold $F_i := \{f \in \mathcal{H} |\ f(A) \cap V_i = \emptyset\} \in \mathcal{F}$, $(1 \leq i \leq n)$.
We have $g(A) \cap V_i \neq \emptyset$, implying $\exists x_g \in A : g(x_g) \in V_i$, so V_i is an open neighbourhood of $g(x_g)$. Thus $\exists F_g \in \mathcal{F} : \forall f \in F_g : f(x_g) \in V_i$, because of the pointwise convergence of $\mathcal{F}$ to g. But now $F_g \cap F_i = \emptyset$ holds - a contradiction to the filter-properties of $\mathcal{F}$. So, $F_0 := \{f \in \mathcal{H} |\ f(A) \notin \bigcup_{i=1}^n V_i\} \in \mathcal{F}$ must hold. Let $V_A := \bigcup_{i=1}^n V_i$, then $\forall f \in F_0 : \exists x_f \in A : f(x_f) \notin V_A$. Thus, a map $\pi : F_0 \to A$ exists, s.t. $\forall f \in F_0 : f(\pi(f)) \notin V_A$. Then $\pi(\mathcal{F})$ is an ultrafilter on A, which must converge to a point $x_0 \in X$ (resp. $x_0 \in A$), because A is relative compact (resp. compact). Because of the pointwise convergence of $\mathcal{F}$ to g, it follows $\mathcal{F}(x_0) \xrightarrow{\sigma} g(x_0)$. From this and $\pi(\mathcal{F}) \xrightarrow{\tau} x_0$ follows $\mathcal{F}(\pi(\mathcal{F})) \xrightarrow{\sigma} g(x_0)$ by the $\mathfrak{A}$-even continuity of $\mathcal{H}$, just meaning

$$\forall V \in \overset{\bullet}{g(x_0)} \cap \sigma : \exists F_V \in \mathcal{F}, A_V \in \pi(\mathcal{F}) : F_V(A_V) \subseteq V \ . \tag{9}$$

On the other hand, $g(\pi(\mathcal{F})) \xrightarrow{\sigma} g(x_0)$ follows from the continuity of g. But $g(\pi(\mathcal{F}))$ is a filter on $g(A)$ and $g(A)$ is closed in the first of the lemma's statements, thus $g(x_0) \in g(A)$ holds, which follows in the second statement directly from $x_0 \in A$. Therefore V_A is an open neighbourhood of $g(x_0)$ and from (9) we get $\exists F_V \in \mathcal{F}, A_V \in \pi(\mathcal{F}) : \forall f \in F_V, a \in A_V : f(a) \in V_A$. But then $F_V \cap \pi^{-1}(A_V) = \emptyset$ and $\pi^{-1}(A_V)$ is a member of $\mathcal{F}$ - a contradiction to the filter-properties of $\mathcal{F}$. So, our assumtion $\mu(\mathcal{F}) \xrightarrow{p} \mu(g)$ must be false. ■

144 Corollary
Let $(X, \tau), (Y, \sigma)$ be topological spaces. Let $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ contain the singletons and consist only of relative compact subsets of X. Let $\mathcal{H} \subseteq C(X, Y)$ be $\mathfrak{A}$-evenly continuous and weakly relative complete in Y^X w.r.t. pointwise convergence and let all members of $\mathfrak{A}$ have closed images under elements of $\mathcal{H}$.
Then $\mu(\mathcal{H})$ is weak relative complete in $\mathfrak{P}_0(Y)^{\mathfrak{A}}$ w.r.t. pointwise convergence, where $\mathfrak{P}_0(Y)$ is equipped with Vietoris topology.

Proof: Let $\mathcal{G}$ be an ultrafilter on $\mu(\mathcal{H})$, which converges pointwise to a function $g \in \mathfrak{P}_0(Y)^{\mathfrak{A}}$. At first, it is clear, that there exists an ultrafilter $\mathcal{F}$ on $\mathcal{H}$, s.t. $\mathcal{G} = \mu(\mathcal{F})$ (corollary 11). From g we derive a function $g' : X \to Y$: for all singletons $\{x\} \in \mathfrak{A}$, we can chose an element y_x from $g(\{x\})$, because the empty set doesn't belong to our range space. Then for each open neighbourhood V of y_x we find $g(\{x\}) \in < V, Y >$, so there must exist a $F \in \mathcal{F}$ with $\forall f \in F : \mu(f)(\{x\}) \in < V, Y >$, just implying $\mathcal{F} \xrightarrow{p} g'$, where g' is chosen s.t. $g' : X \to Y : g'(x) := y_x \in g(\{x\})$. Now, because of the weak relative completeness of $\mathcal{H}$, there must exist a function $g_1 \in \mathcal{H}$ with

$\mathcal{F} \xrightarrow{p} g_1$. From lemma 143 follows $\mu(\mathcal{F}) = \mathcal{G} \xrightarrow{p} \mu(g_1) \in \mu(\mathcal{H})$. ■

145 Corollary
Let $(X,\tau),(Y,\sigma)$ be topological spaces, Y Hausdorff. Let $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ contain the singletons and consist only of compact subsets of X. Let $\mathcal{H} \subseteq C(X,Y)$ be $\mathfrak{A}$-evenly continuous and weakly relative complete in Y^X w.r.t. pointwise convergence.
Then $\mu(\mathcal{H})$ is closed in $K(Y)^{\mathfrak{A}}$ w.r.t. pointwise convergence, where $K(Y)$ is equipped with Vietoris topology.

Proof: Of course, compact subsets are relative compact. Continuous images of compact sets are compact and therefore closed in the Hausdorff-space Y. So, corollary 144 applies, yielding $\mu(\mathcal{H})$ to be weakly relative complete in $\mathfrak{P}_0(Y)^{\mathfrak{A}}$ and consequently in $K(Y)^{\mathfrak{A}}$ (since $K(Y)^{\mathfrak{A}}$ is a subspace of $\mathfrak{P}_0(Y)^{\mathfrak{A}}$ w.r.t. pointwise convergence). But if Y is Hausdorff, then $K(Y)$ with Vietoris-topology is, and consequently, the function space is Hausdorff, too. So, by proposition 114, $\mu(\mathcal{H})$ is closed. ■

Note, that this is somewhat other than Mizokami showed. We require the additional condition of $\mathfrak{A}$-even continuity and get the stronger result of closedness in $\mathfrak{P}_0(Y)^{\mathfrak{A}}$, not only in $C(\mathfrak{A}, C_Y(\mathfrak{A}))$ - because we will need it.

146 Corollary
Let $(X,\tau),(Y,\sigma)$ be topological spaces, Y Hausdorff and T_3. Let $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ contain the singletons and consist only of compact subsets of X. Let $\mathcal{H} \subseteq C(X,Y)$ be evenly continuous and weakly relative complete in $C(X,Y)$ w.r.t. pointwise convergence. Then $\mu(\mathcal{H})$ is closed in $K(Y)^{\mathfrak{A}}$.

Proof: If an ultrafilter $\mathcal{F}$ on $\mathcal{H}$ converges pointwise in Y^X to a function g, then from the even continuity of $\mathcal{H}$ follows, that $\mathcal{F}$ converges continuously to g and then with theorem 30 in [2] from T_3 the continuity of g. So, $\mathcal{F}$ converges in $C(X,Y)$ and therefore in $\mathcal{H}$, because of the weak relative completeness in $C(X,Y)$. Thus, $\mathcal{H}$ is indeed weak relative complete in Y^X and corollary 145 applies. ■

147 Theorem
Let $(X,\tau),(Y,\sigma)$ be topological spaces and let $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ contain the singletons. Then a set $\mathcal{H} \subseteq Y^X$ is relative compact in $(Y^X, \tau_{\mathfrak{A}})$ if and only if

(1) For all ultrafilters $\mathcal{F}$ on $\mathcal{H}$ with $\mathcal{F} \xrightarrow{p} f \in Y^X$ exists a function $g \in Y^X$, s.t. $\mu(\mathcal{F}) \xrightarrow{p} \mu(g) \in \mathfrak{P}_0(Y)^{\mathfrak{A}}$, where $\mathfrak{P}_0(Y)$ is equipped with Vietoris topology, and

(2) for all $A \in \mathfrak{A}$, the family $\mu(\mathcal{H})(A) := \{f(A) |\ f \in \mathcal{H}\}$ is relative compact in $\mathfrak{P}_0(Y)$ w.r.t. Vietoris topology.

Proof: Because $\mathfrak{A}$ contains the singletons, the Mizokami-map $\mu : (\mathcal{H}, \tau_{\mathfrak{A}}) \to (\mu(\mathcal{H}), \tau_p)$ is continuous, open and bijective by lemma 137. Now, $(\mathfrak{P}_0(Y)^{\mathfrak{A}}, \tau_p)$ is naturally isomorphic to $\prod_{A\in\mathfrak{A}} \mathfrak{P}_0(Y)_A$ with Tychonoff product topology, where all $\mathfrak{P}_0(Y)_A$ are clones of $\mathfrak{P}_0(Y)$ (see [39],2.2), let

$$\pi : (\mathfrak{P}_0(Y)^{\mathfrak{A}}, \tau_p) \to \prod_{A\in\mathfrak{A}} \mathfrak{P}_0(Y)_A : f \to (f(A))_{A\in\mathfrak{A}}$$

be the isomorphism. Then $\pi(\mu(\mathcal{H}))$ is just a subset of the product $\prod_{A\in\mathfrak{A}} \mu(\mathcal{H})(A)$. Let (1) and (2) be fulfilled. Then all $\mu(\mathcal{H})(A)$ are relative compact in $\mathfrak{P}_0(Y)$ by (2), so the product $\prod_{A\in\mathfrak{A}} \mu(\mathcal{H})(A)$ is relative compact in $\prod_{A\in\mathfrak{A}} \mathfrak{P}_0(Y)_A$ by the Tychonoff-theorem for relative compact subsets (see 1.44 in [39]). Thus, as a subset of a relative compact set, $\pi(\mu(\mathcal{H}))$ itself is relative compact in $\prod_{A\in\mathfrak{A}} \mathfrak{P}_0(Y)_A$. Let $\mathcal{F}$ be an ultrafilter on $\mathcal{H}$, then $\pi(\mu(\mathcal{F}))$ is an ultrafilter on $\pi(\mu(\mathcal{H}))$, which now must converge in $\prod_{A\in\mathfrak{A}} \mathfrak{P}_0(Y)_A$, implying $\mu(\mathcal{F})$ converges pointwise to a function $f \in \mathfrak{P}_0(Y)^{\mathfrak{A}}$, by isomorphism. Then by proposition 138, $\mathcal{F}$ converges pointwise to a function $f' \in Y^X$. From (1) now follows the existence of a function $g \in Y^X$ with $\mu(\mathcal{F}) \xrightarrow{p} \mu(g)$ and thus $\mathcal{F} \xrightarrow{\tau_{\mathfrak{A}}} g$, because the Mizokami-map is open between $(Y^X, \tau_{\mathfrak{A}})$ and $(\mu(Y^X), \tau_p)$, by lemma 137.
If otherwise $\mathcal{H}$ is relative compact in Y^X w.r.t. $\tau_{\mathfrak{A}}$, then every ultrafilter $\mathcal{F}$ on $\mathcal{H}$ $\tau_{\mathfrak{A}}$-converges to a function $g \in Y^X$, and therefore $\mu(\mathcal{F})$ converges pointwise to $\mu(g)$ by the continuity of the Mizokami-map, and of course, $\mathcal{F}$ converges pointwise to g, because $\mathfrak{A}$ contains the singletons - so, (1) is fulfilled. Furthermore, an ultrafilter $\mathcal{G}$ on $\mu(\mathcal{H})(A)$ induces an ultrafilter $\mathcal{G}'$ on $\mu(\mathcal{H})$, whose evaluation on A is just $\mathcal{G}$, by corollary 11, and therefore an ultrafilter $\mathcal{F}$ on $\mathcal{H}$ exists, with $\mu(\mathcal{F}) = \mathcal{G}'$, by bijectivity of the Mizokami-map. Now, $\mathcal{F}$ $\tau_{\mathfrak{A}}$-converges to a function $f \in Y^X$, by the relative compactness of $\mathcal{H}$, thus $\mu(\mathcal{F})(A) = \mathcal{G}$ converges to $\mu(f)(A)$, because of the continuity of the Mizokami-map - so, (2) is fulfilled. ■

148 Corollary
Let $(X, \tau), (Y, \sigma)$ be topological spaces and let $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ contain the singletons. Then a set $\mathcal{H} \subseteq Y^X$ is relative compact in $(Y^X, \tau_{\mathfrak{A}})$, if

(1) For all ultrafilters $\mathcal{F}$ on $\mathcal{H}$ with $\mathcal{F} \xrightarrow{p} f \in Y^X$ exists a function $g \in Y^X$, s.t. $\mu(\mathcal{F}) \xrightarrow{p} \mu(g) \in \mathfrak{P}_0(Y)^{\mathfrak{A}}$, where $\mathfrak{P}_0(Y)$ is equipped with Vietoris topology, and

(2) for all $A \in \mathfrak{A}$, the set $\mathcal{H}(A) := \bigcup_{f\in\mathcal{H}} f(A)$ is relative compact in Y.

Proof: If $\mathcal{H}(A)$ is relative compact in Y, then $\mathfrak{P}_0(\mathcal{H}(A))$ is in $\mathfrak{P}_0(Y)$ w.r.t. Vietoris topology, by lemma 117, thus the subset $\mu(\mathcal{H})(A)$ is, and then the theorem 147 applies. ■

149 Corollary
Let $(X,\tau),(Y,\sigma)$ be topological spaces and let $\mathfrak{A}\subseteq\mathfrak{P}_0(X)$ consist only of relative compact subsets of X and contain the singletons. Let $\mathcal{H}\subseteq C(X,Y)$ have the following properties:

(1) $\mathcal{H}$ is weakly relative complete in Y^X w.r.t. pointwise convergence,

(2) $\mathcal{H}$ is $\mathfrak{A}$-evenly continuous,

(3) the images of all members of $\mathfrak{A}$ under elements of $\mathcal{H}$ are closed in Y and

(4) for all $A\in\mathfrak{A}$, each ultrafilter φ on A converges to a point $x_0\in X$, s.t. $\mathcal{H}(x_0):=\{f(x_0)|\ f\in\mathcal{H}\}$ is relative compact in Y.

Then $\mathcal{H}$ is compact w.r.t. $\tau_{\mathfrak{A}}$.
If otherwise $\mathcal{H}$ is compact w.r.t. $\tau_{\mathfrak{A}}$, then (1) follows and for all $A\in\mathfrak{A}$ is $\mathcal{H}(A):=\bigcup_{f\in\mathcal{H}}f(A)$ relative compact in Y.

Proof: Condition (1) ensures, that every ultrafilter $\mathcal{F}$ on $\mathcal{H}$, which pointwise converges in Y^X, converges in $\mathcal{H}$, too. From (2) and (3) follows, that for each ultrafilter $\mathcal{F}$ on $\mathcal{H}$ always $\mathcal{F}\xrightarrow{p}g\in C(X,Y)$ implies $\mu(\mathcal{F})\xrightarrow{p}\mu(g)$, by lemma 143(1). From (2) and (4) follows the relative compactness of all $\mathcal{H}(A)$ for $A\in\mathfrak{A}$, by lemma 142. Thus, corollary 148 applies, yielding the relative compactness of $\mathcal{H}$ in Y^X. Now, from (1) and proposition 114 follows the compactness.
If otherwise $\mathcal{H}$ is compact w.r.t. $\tau_{\mathfrak{A}}$, then it is compact w.r.t. pointwise convergence, too, and so (1) follows by proposition 114, and the relative compactness of all $\mathcal{H}(A), A\in\mathfrak{A}$ follows by corollary 121, because $\mu(\mathcal{H})(A)$ is compact w.r.t. the Vietoris topology by the continuity of both, the Mizolami-map and the projections $p_A:\mathfrak{P}_0(Y)^{\mathfrak{A}}\to\mathfrak{P}_0(Y):g\to g(A)$. ■

150 Corollary
Let $(X,\tau),(Y,\sigma)$ be topological spaces and let $\mathfrak{A}\subseteq\mathfrak{P}_0(X)$ consist only of compact subsets of X and contain the singletons. Let $\mathcal{H}\subseteq C(X,Y)$ have the following properties:

(1) $\mathcal{H}$ is weakly relative complete in Y^X w.r.t. pointwise convergence,

(2) $\mathcal{H}$ is $\mathfrak{A}$-evenly continuous,

(3) for all $A\in\mathfrak{A}$, each ultrafilter φ on A converges to a point $x_0\in X$, s.t. $\mathcal{H}(x_0):=\{f(x_0)|\ f\in\mathcal{H}\}$ is relative compact in Y.

Then $\mathcal{H}$ is compact w.r.t. $\tau_{\mathfrak{A}}$.
If otherwise $\mathcal{H}$ is compact w.r.t. $\tau_{\mathfrak{A}}$, then (1) follows and for all $A\in\mathfrak{A}$ is $\mathcal{H}(A):=\bigcup_{f\in\mathcal{H}}f(A)$ compact in Y.

Proof: Copy the proof of corollary 149, but use part (2) of lemma 143, instead of part (1), then the closedness of the images is not needed. ■

Note, that all requirements, in order to make $\mathcal{H}$ compact, are focused to $\mathcal{H}$ and $\mathfrak{A}$. There is no condition concerning the spaces X, Y (except, that they should be topological spaces). This seems to be natural, because in fact, the compactness of $\mathcal{H}$ is in question, not the compactness of X or Y. But, of course, special properties of the range space may simplify the requirements, as the following shows.

151 Corollary
Let $(X,\tau),(Y,\sigma)$ be topological spaces, Y Hausdorff. Then a set of functions $\mathcal{H} \subseteq C(X,Y)$ is compact w.r.t. the compact-open topology τ_{co}, if and only if it has the following properties:

(1) $\mathcal{H}$ is closed in Y^X w.r.t. pointwise convergence,

(2) $\mathcal{H}$ is evenly continuous on all compact subsets and

(3) for all $A \in K(X)$ is $\mathcal{H}(A) := \bigcup_{f\in\mathcal{H}} f(A)$ compact in Y.

Proof: Let $\mathfrak{A} := K_0(X)$, the set of all nonempty compact subsets of X, so $\tau_{\mathfrak{A}}$ is just the compact-open topology τ_{co}. Because Y is Hausdorff, from (2) we get the $\mathfrak{A}$-even continuity of $\mathcal{H}$, by proposition 140, so, if (1), (2), (3) are fulfilled, corollary 150 applies, yielding $\mathcal{H}$ to be compact w.r.t. τ_{co}.
If otherwise $\mathcal{H}$ is compact w.r.t. τ_{co}, we get (1) and (3) from corollary 150 again, and (2) from proposition 141. ■

To require closedness of $\mathcal{H}$ here, instead of weak relative completeness as in corollary 150, is not really stronger, because Y^X is Hausdorff w.r.t. pointwise convergence, whenever Y is, and so closedness and weak relative completeness coincide by proposition 114. This corollary is just a repaired version of Edwards' statement 3.13 in [15], where only closedness of $\mathcal{H}$ in $C(X,Y)$ - not in Y^X - is required and condition 151(2) is omitted. The following shows, that this is indeed not enough to get compactness for $\mathcal{H}$.

152 Example: Let the interval $[0,1] := \{x \in \mathbb{R} |\ 0 \le x \le 1\} \subseteq \mathbb{R}$ be equipped with euclidian topology,

$$\begin{aligned} p_0 : [0,1] \to [0,1] \quad &: \quad p_0(x) = 0 \text{ and} \\ p_r : [0,1] \to [0,1] \quad &: \quad p_r(x) = x^r, \text{ for } r \in \mathbb{R}, r \ge 1 \end{aligned}$$

and $\mathcal{H}_1 := \{p_r |\ r \in \mathbb{R}, r \ge 1\}$. If $K \subseteq [0,1]$ is compact, then $\mathcal{H}_1(K) := \bigcup_{f\in\mathcal{H}_1} f(K)$ is compact, too. Moreover, $\mathcal{H}_1$ is closed in $C([0,1],[0,1])$ w.r.t. the pointwise topology. (But, of course, it is not closed in $[0,1]^{[0,1]}$.) So, the assertions of Edwards' statement are fulfilled, but $\mathcal{H}_1$ fails to be compact w.r.t. the compact-open topology

- in fact, it is not even compact w.r.t. the pointwise topology, because there is the simple filter $[\{\{p_k|\ k \geq n\}|\ n \in I\!N\}]$ on $\mathcal{H}_1$, which pointwise converges in $[0,1]^{[0,1]}$ to the function

$$q : [0,1] \to [0,1] : q(x) := \begin{cases} 0 & ; \quad x < 1 \\ 1 & ; \quad x = 1 \end{cases}$$

implying, that all refining ultrafilters converge to this function, too. So, they all fail to converge in $\mathcal{H}_1$, because $[0,1]^{[0,1]}$ is Hausdorff.

Proof: Let $K \subseteq [0,1]$ be compact. Then K contains a maximal element x_{max}, if K is not empty. We now have two cases:

(1) $x_{max} < 1$
Then $\forall y \in \mathcal{H}_1(K) : \exists x \in K, r \in [1,\infty) : y = x^r \leq x \leq x_{max}$ holds, implying $\mathcal{H}_1(K) \subseteq [0, x_{max}]$.
Otherwise we have $\forall y \in (0, x_{max}] : r := \log_{x_{max}} y \geq 1$, thus $p_r \in \mathcal{H}_1$ and consequently $y = p_r(x_{max}) \in \mathcal{H}_1(K)$. Now, $0 \in \mathcal{H}_1(K)$ always holds for nonempty K, because of p_0. So, we find $[0, x_{max}] \subseteq \mathcal{H}_1(K)$, yielding now $\mathcal{H}_1(K) = [0, x_{max}]$, being compact.

(2) $x_{max} = 1$

(a) 1 is not an accumulation-point of K.
Then $K \setminus \{1\}$ is compact, too, and has (if not empty) a maximal element $x'_{max} < 1$. For the same reasons as above, we get $\mathcal{H}_1(K \setminus \{1\}) = [0, x'_{max}]$ and thus $\mathcal{H}_1(K) = [0, x'_{max}] \cup \{1\}$, being compact.

(b) 1 is an accumulation point of K.
Then $1 \in K$ holds, because K is compact and $[0,1]$ is Hausdorff. So, $1 \in \mathcal{H}_1(K)$ is ensured, too.
Moreover, we have $\forall y \in (0,1) : \exists x \in K : y \leq x$, implying $y = p_r(x) \in \mathcal{H}_1(K)$ with $r := \log_x y \geq 1$. Thus $(0,1) \subseteq \mathcal{H}_1(K)$, yielding $\mathcal{H}_1(K) = [0,1]$, being compact.

So, in any case, $\mathcal{H}_1(K)$ is compact, whenever K is.

Now we have to show, that $\mathcal{H}_1$ is closed in $C([0,1],[0,1])$ w.r.t. the pointwise convergence.

Let φ be an ultrafilter on $\mathcal{H}_1$, pointwise converging to a function $f \in [0,1]^{[0,1]}$, but is not the singleton-filter $\dot{p_0}$ (If $\varphi = \dot{p_0}$, it converges obviously only to $p_0 \in \mathcal{H}_1$). It is then clear, that $\varphi(0) = \dot{0} \to 0$ and $\varphi(1) = \dot{1} \to 1$ hold, so by the Hausdorffness of

$[0,1]$, we get $f(0)=0$ and $f(1)=1$.
There is a "projection map" $\pi : \mathcal{H}_1 \to [1,\infty) : \pi(p_r) := r$.
Now, $\pi(\varphi)$ is an ultrafilter on $[1,\infty)$ and we have two cases:

(1) All members of $\pi(\varphi)$ are unbounded.
Then we find $f(x)=0$ for all $x \in (0,1)$:
Assume $f(x)>0$.
Then there exists ε with $0<\varepsilon<f(x)$ and we have $\forall M \in \varphi : \exists r \in M : r > log_{f(x)}(\frac{\varepsilon}{3}) > 1$, implying $M(x) \cap [0,\frac{\varepsilon}{3}) \neq \emptyset$, thus $(f(x)-\frac{\varepsilon}{3}, f(x)+\frac{\varepsilon}{3}) \not\supseteq M(x)$ and so $(f(x)-\frac{\varepsilon}{3}, f(x)+\frac{\varepsilon}{3}) \notin \varphi(x)$, yielding $\varphi(x) \not\to f(x)$ - in contradiction to the pointwise convergence of φ. Thus $f(x)=0$ must hold for all $x \in (0,1)$, $f(0)=0, f(1)=1$ and therefore $f=q \notin C([0,1],[0,1])$.

(2) There exists $M \in \varphi$ with $\pi(M)$ is bounded.
Then $\pi(M)$ is relative compact in $[1,\infty)$, so the ultrafilter $\pi(\varphi)$, containing $\pi(M)$, converges to a point $r_0 \in [1,\infty)$. This means $\forall \varepsilon > 0 : \exists M_\varepsilon \in \varphi : \forall r \in \pi(M) : |r-r_0| < \varepsilon$. So, for all $x \in (0,1)$ and $0<\delta<x^{r_0}$ we can chose $\varepsilon_{x,\delta} := \min\{\log_x(1-\frac{\delta}{x^{r_0}}), -\log_x(1+\frac{\delta}{x^{r_0}})\}$ and find $\exists M_{\varepsilon_{x,\delta}} \in \varphi : \forall p_r \in M : |p_r(x)-p_{r_0}(x)| = |x^r - x^{r_0}| < \delta$, implying $\varphi(x) \to x^{r_0} = p_{r_0}(x)$. Thus, φ converges pointwise to $p_{r_0} \in \mathcal{H}_1$ and only to this function, because $[0,1]^{[0,1]}$ is Hausdorff w.r.t. the pointwise convergence.

All in all, if φ converges to a continuous function, then this function belongs to $\mathcal{H}_1$, so $\mathcal{H}_1$ is closed in $C([0,1],[0,1])$.

We will give an additional example, to show, that non-closedness in Y^X w.r.t. pointwise convergence is not the essential reason for a set $\mathcal{H}$ of continuous functions to be non-compact - but the absence of additional properties, like some kind of even continuity, for example.

153 Example: Let $X=[0,1] \subseteq I\!R$ be equipped with euclidian topology, $Y=[0,1]$ with euclidian topology, too. Now, let

$$c_s \;:\; X \to Y : c_s(x) = s,\ s \in [0,1]$$

and let $\mathcal{H}_2 := \{c_s |\ 0 \le s \le 1\}$. Furthermore, let

$$w_n : X \to Y : w_n(x) = \begin{cases} 0 & ; \quad 0 \le x \le \frac{1}{3n} \\ 3nx-1 & ; \quad \frac{1}{3n} < x \le \frac{2}{3n} \\ -3nx+3 & ; \quad \frac{2}{3n} < x \le \frac{1}{n} \\ 0 & ; \quad \frac{1}{n} < x \le 1 \end{cases}$$

with $n \in I\!N, n \ge 2$ and then let $\mathcal{H}_3 := \{w_n |\ n \in I\!N, n \ge 2\}$.
Then $\mathcal{H} := \mathcal{H}_2 \cup \mathcal{H}_3$ is closed in Y^X w.r.t. pointwise convergence and for all subsets

K (especially for all compact subsets) of X is $\mathcal{H}(K)$ compact. But $\mathcal{H}$ is not compact w.r.t. the compact-open topology.

Proof: It is clear, that $\mathcal{H}_2(K) = [0,1]$ for all nonempty subsets K of X. So, in any case $\mathcal{H}_3(K) \subseteq \mathcal{H}_2(K)$ and consequently $\mathcal{H}(K) = \mathcal{H}_2(K) \cup \mathcal{H}_3(K) = \mathcal{H}_2(K)$ is compact.
To see, that $\mathcal{H}$ is closed in Y^X, let $\mathcal{F}$ be an ultrafilter on $\mathcal{H}$, which converges pointwise to a function $g \in Y^X$. Then $\mathcal{F}$ either contains $\mathcal{H}_2$ or $\mathcal{H}_3$, because it is an ultrafilter. If $\mathcal{F}$ contains $\mathcal{H}_2$, then it's evaluation filter on every point of X is the same - and as an ultrafilter in the compact Y this converges to a point of Y, thus $\mathcal{F}$ converges pointwise to the associated constant function. If $\mathcal{F}$ contains $\mathcal{H}_3$, then either $\mathcal{F}$ is a singleton-filter (and therefore converges pointwise to its generating element of $\mathcal{H}_3$) or it contains the filter $\mathcal{G} := [\{\{w_k|\ k \geq n\}|\ n \in I\!N, n \geq 2\}]$. But this filter obviously converges pointwise to $c_0 \in \mathcal{H}$, and so any refining ultrafilter does.
Thus, $\mathcal{F}$ converges in $\mathcal{H}$, whenever it converges in Y^X, so $\mathcal{H}$ is closed in Y^X w.r.t. pointwise convergence, because Y^X is Hausdorff.
Otherwise, just the filter $\mathcal{G}$ fails to converge w.r.t. the compact-open topology τ_{co}: the convergence w.r.t. τ_{co} coincides with continuous convergence, because X is locally compact. The only function, to which $\mathcal{G}$ could converge w.r.t. τ_{co} is c_0, because it converges pointwise only to this function. So, for the neighbourhood-filter $\underline{U}(0)$ of zero, $\mathcal{G}(\underline{U}(0))$ should converge to 0 - but it doesn't, because for any $G \in \mathcal{G}$ and any open neighbourhood U of 0 we find $1 \in G(U)$. Thus, there must exist a refining ultrafilter of $\mathcal{G}$, which doesn't $\tau_{co}-$converge to c_0 and therefore completely fails to converge w.r.t. τ_{co}. ■

154 Corollary
Let $(X,\tau),(Y,\sigma)$ be topological spaces, Y Hausdorff. Then a set of functions $\mathcal{H} \subseteq C(X,Y)$ is compact w.r.t. the compact-open topology τ_{co}, if and only if it has the following properties:

(1) $\mathcal{H}$ is closed in Y^X w.r.t. pointwise convergence,

(2) $\mathcal{H}$ is evenly continuous on all compact subsets and

(3) for all $x \in X$ is $\mathcal{H}(x) := \{f(x)|\ f \in \mathcal{H}\}$ relative compact in Y.

Proof: Follows directly from corollary 151 and lemma 142. ■

Our last thing to do in this section, is to give a Mizokami-like mapping theorem, concerning the structure of continuous convergence instead of compact-open topology. We will map the function space $(C(X,Y), q_c)$ into the function space $(C(\mathfrak{C}(X),\mathfrak{C}(Y)),\ q_p)$, where $\mathfrak{C}(X),\mathfrak{C}(Y)$ are endowed with the Vietoris-pseudotopologies. The map is of the same natural kind as before, but should be studied a little

more here, before it is applied:

$$\mu : C(X,Y) \to \mathfrak{C}(Y)^{\mathfrak{C}(X)} : f \to \mu(f) : \mu(f)(\varphi) := f(\varphi)$$

Here, proposition 91 ensures, that we really map into $\mathfrak{C}(Y)^{\mathfrak{C}(X)}$, not only into $\mathfrak{F}(Y)^{\mathfrak{C}(X)}$.

155 Proposition
Let $(X,\tau),(Y,\sigma)$ be topological spaces. Then for the map

$$\mu : C(X,Y) \to \mathfrak{C}(Y)^{\mathfrak{C}(X)} : f \to \mu(f) : \mu(f)(\varphi) := f(\varphi)$$

hold

(1) *μ is injective,*

(2) *$\forall \mathcal{F} \in \mathfrak{F}(C(X,Y)), \varphi \in \mathfrak{C}(X) : (\mu(\mathcal{F})(\varphi))^{\cup\cap} \supseteq \mathcal{F}(\varphi)$ and*

(3) *$\forall \Phi \in \mathfrak{F}(\mathfrak{C}(X)), f \in C(X,Y) : f(\Phi^{\uparrow}) = (\mu(f)(\Phi))^{\uparrow}$.*

Proof: (1) follows directly from the fact, that the singleton-filters are compactoid. So, if $\mu(f) = \mu(g)$, especially $\forall x \in X : \mu(f)(\overset{\bullet}{x}) = \mu(g)(\overset{\bullet}{x})$ and therefore $\forall x \in X : f(x) = g(x)$ holds.
(2): $M \in \mathcal{F}(\varphi) \Leftrightarrow \exists F \in \mathcal{F}, P \in \varphi : \forall g \in F : g(P) \subseteq M \Rightarrow \exists F \in \mathcal{F} : \forall g \in F : \exists P_g \in \varphi : g(P_g) \subseteq M \Leftrightarrow \exists F \in \mathcal{F} : M \in \bigcup_{g \in F} g(\varphi) \Leftrightarrow M \in (\mu(\mathcal{F})(\varphi))^{\cup\cap}$.
(3): We have

$$\begin{aligned}
\mathfrak{M} \in f(\Phi^{\uparrow}) \quad \Leftrightarrow \exists \mathfrak{A} \in \Phi : \quad & f(\mathfrak{F}_0(\bigcap_{\chi \in \mathfrak{A}} \chi)) \subseteq \mathfrak{M} \\
& \Leftrightarrow \mathfrak{F}_0(f(\bigcap_{\chi \in \mathfrak{A}} \chi)) \subseteq \mathfrak{M} \text{ (by proposition 12)} \\
& \Leftrightarrow \mathfrak{F}_0(\bigcap_{\chi \in \mathfrak{A}} f(\chi)) \subseteq \mathfrak{M} \text{ (by proposition 3)} \\
& \Leftrightarrow \mathfrak{M} \in (\mu(f)(\Phi))^{\uparrow} \quad \blacksquare
\end{aligned}$$

156 Lemma
Let $(X,\tau),(Y,\sigma)$ be topological spaces. Then with the map

$$\mu : C(X,Y) \to \mathfrak{C}(Y)^{\mathfrak{C}(X)} : f \to \mu(f) : \mu(f)(\varphi) := f(\varphi)$$

holds, that $\mu(f)$ is continuous w.r.t. $q'_V(\tau), q'_V(\sigma)$ for all $f \in C(X,Y)$.

Proof: Let $f \in C(X,Y)$ and $\Phi \in \mathfrak{F}_0(\mathfrak{C}(X)), \varphi \in \mathfrak{C}(X)$ with $(\Phi,\varphi) \in q'_V(\tau)$ be given. For every $\psi_0 \in \mathfrak{F}_0(f(\varphi))$ there is a $\varphi_0 \in \mathfrak{F}_0(\varphi)$ with $f(\varphi_0) = \psi_0$, by corollary 11. Because Φ converges to φ w.r.t. $q'_l(\tau)$, we know, that there is a $\Phi_1 \in \mathfrak{F}_0(\Phi^{\uparrow})$ such that every member A of φ contains an element a, s.t. φ_0 and

$\Phi_1^{\bigcup}$ both converge to a w.r.t. τ. Thus every member of $f(\varphi)$ contains an element $f(a)$ s.t. $\psi = f(\varphi_0)$ and $f(\Phi_1^{\bigcup})$ both converge to $f(a)$, because of the continuity of f. But then $f(\Phi_1)^{\bigcup}$ converges to $f(a)$ because of proposition 122(2), and we know $f(\Phi_1) \in f(\mathfrak{F}_0(\Phi^\uparrow)) = \mathfrak{F}_0(f(\Phi^\uparrow)) = \mathfrak{F}_0(\mu(f)(\Phi)^\uparrow)$ from the propositions 12 and 155(3). So we find $(\mu(f)(\Phi), \mu(f)(\varphi)) \in q_l'(\sigma)$.
Furthermore, for every $\Phi_2 \in \mathfrak{F}_0(\mu(f)(\Phi)^\uparrow)$ we observe $\mathfrak{F}_0(\mu(f)(\Phi)^\uparrow) = \mathfrak{F}_0(f(\Phi^\uparrow)) = f(\mathfrak{F}_0(\Phi^\uparrow))$ because of the propositions 155(3) and 12, and conclude, that there exists $\Phi_1 \in \mathfrak{F}_0(\Phi^\uparrow)$ with $f(\Phi_1) = \Phi_2$. Now, Φ converges to φ w.r.t. $q_u'(\tau)$, so there exists $\varphi_0 \in \mathfrak{F}_0(\varphi)$ s.t. every $A \in \varphi$ contains an element a, to which φ_0 and $\Phi_1^{\bigcup}$ both converge w.r.t. τ. Thus every $f(A) \in f(\varphi)$ contains an element $f(a)$ s.t. $f(\Phi_1^{\bigcup})$ and $f(\varphi_0) \in \mathfrak{F}_0(f(\varphi))$ both converge to $f(a)$ w.r.t. σ, because of the continuity of f. Now, from proposition 122(2) it follows, that $f(\Phi_1)^{\bigcup} = \Phi_2^{\bigcup}$ converges to $f(a)$, too. So we find $(\mu(f)(\Phi), \mu(f)(\varphi)) \in q_u'(\sigma)$, implying now $(\mu(f)(\Phi), \mu(f)(\varphi)) \in q_V'(\sigma)$ because of the above proven q_l'-convergence, and therefore, because this holds for all $(\Phi, \varphi) \in q_V'(\tau)$, the continuity of $\mu(f)$ follows. ■

157 Lemma
Let $(X, \tau), (Y, \sigma)$ be topological spaces. Then the map

$$\mu : C(X, Y) \to C(\mathfrak{C}(X), \mathfrak{C}(Y)) : f \to \mu(f) : \varphi \to f(\varphi)$$

is continuous and injective, where $C(X, Y)$ is endowed with the structure q_c of continuous convergence, $C(\mathfrak{C}(X), \mathfrak{C}(Y))$ with the structure q_p of pointwise convergence, for $\mathfrak{C}(X)$ and $\mathfrak{C}(Y)$ being equipped with the Vietoris-pseudotopologies $q_V'(\tau)$ and $q_V'(\sigma)$, respectively.

Proof: By proposition 155 we know, that $\mu()$ is injective and lemma 156 says, that $\mu()$ maps $C(X, Y)$ into $C(\mathfrak{C}(X), \mathfrak{C}(Y))$. To prove continuity of μ, let $\mathcal{F} \in \mathfrak{F}(C(X, Y)), f \in C(X, Y)$ with $(\mathcal{F}, f) \in q_c$ and an arbitrary $\varphi \in \mathfrak{C}(X)$ be given. Then for all $\psi_0 \in \mathfrak{F}_0(f(\varphi))$, by corollary 11 there exists a $\varphi_0 \in \mathfrak{F}_0(\varphi)$ such that $f(\varphi_0) = \psi_0$. Now, we have naturally $\forall B \in f(\varphi) : \exists A_B \in \varphi : f(A_B) \subseteq B$, and because of the compactoidness of φ we know $\exists a \in A_B : (\varphi_0, a) \in q_\tau$. By the continuity of f we get now $(f(\varphi_0), f(a)) \in q_\sigma$, i.e. $(\psi_0, f(a)) \in q_\sigma$. Because of the continuous convergence of $\mathcal{F}$ to f, we find $(\mathcal{F}(\varphi_0), f(a)) \in q_\sigma$.
Observe now, that $\mu(\mathcal{F})(\varphi_0)$ is a filter on $\mathfrak{F}_0(Y)$, which refines $\mu(\mathcal{F})(\varphi)^\uparrow$, because φ_0 is an ultrafilter and it refines φ. This yields $\mathfrak{F}_0(\mu(\mathcal{F})(\varphi_0)) \subseteq \mathfrak{F}_0(\mu(\mathcal{F})(\varphi)^\uparrow)$.
So, let $\Phi_1 \in \mathfrak{F}_0(\mu(\mathcal{F})(\varphi_0))$. Then $\Phi_1^{\bigcup} \supseteq (\mu(\mathcal{F})(\varphi_0))^{\bigcup}$, implying $\Phi_1^{\bigcup} \supseteq \mathcal{F}(\varphi_0)$ by proposition 155(2), thus $\Phi_1^{\bigcup}$ converges to $f(a)$, because $\mathcal{F}(\varphi_0)$ does. All in all, every $B \in f(\varphi)$ contains an element $b = f(a)$ to which both, ψ_0 and $\Phi_1^{\bigcup}$, converge. This holds for all $\psi \in \mathfrak{F}_0(\varphi)$, implying $(\mu(\mathcal{F})(\varphi), \mu(f)(\varphi)) \in q_l'(\sigma)$.
Furthermore, let $\Phi_1 \in \mathfrak{F}_0(\mu(\mathcal{F})(\varphi)^\uparrow)$. Then $\Phi_1^{\bigcup}$ is an ultrafilter on Y by proposition 122. Thus, the collection $\mathfrak{B} := \{O \in \sigma |\ O \notin \Phi_1^{\bigcup}\}$ is closed under finite unions because of proposition 7. Assume now, that every refining ultrafilter of $f(\varphi)$ would

contain an element of $\mathfrak{B}$. Then by lemma 9, the filter $f(\varphi)$ itself must contain an open set, which doesn't belong to $\Phi_1^{\sqcup\!\sqcap}$. But proposition 155(2) ensures $\Phi_1^{\sqcup\!\sqcap} \supseteq \mathcal{F}(\varphi)$ and by lemma 92 we know, that $\mathcal{F}$ converges $\mathfrak{C}(X)$-continuously to f, just yielding $\mathcal{F}(\varphi) \supseteq f(\varphi) \cap \sigma$ - a contradiction. Thus, there must exist an refining ultrafilter ψ of $f(\varphi)$, whose open members are all contained in $\Phi_1^{\sqcup\!\sqcap}$, too, so $\Phi_1^{\sqcup\!\sqcap}$ converges to the same points as ψ does, and consequently $(\mu(\mathcal{F})(\varphi), \mu(f)(\varphi)) \in q'_u(\sigma)$ holds, yielding $(\mu(\mathcal{F})(\varphi), \mu(f)(\varphi)) \in q'_V(\sigma)$, because of the result above. These convergence relations are valid for all $\varphi \in \mathfrak{C}(X)$, so $(\mu(\mathcal{F}), \mu(f)) \in q_p$ follows. ■

158 Theorem
Let $(X,\tau),(Y,\sigma)$ be topological spaces and $\mathcal{H}$ an evenly continuous subset of $C(X,Y)$. Then the map

$$\mu : \mathcal{H} \to C(\mathfrak{C}(X), \mathfrak{C}(Y)) : f \to \mu(f) : \varphi \to f(\varphi)$$

is continuous, injective and its inverse map from $\mu(\mathcal{H})$ to $\mathcal{H}$ is continuous, too, where $\mathcal{H}$ is endowed with the structure q_c of continuous convergence, $C(\mathfrak{C}(X), \mathfrak{C}(Y))$ with the structure q_p of pointwise convergence, for $\mathfrak{C}(X)$ and $\mathfrak{C}(Y)$ being equipped with the Vietoris-pseudotopologies $q'_V(\tau)$ and $q'_V(\sigma)$, respectively.

Proof: According to lemma 157, we have only to show, that the inverse map is continuous. So, let $\mathcal{F} \in \mathfrak{F}_0(\mathcal{H})$ with $\mu(\mathcal{F}) \xrightarrow{p} \mu(f) \in \mu(\mathcal{H})$ be given.
Because all singleton-filter $\overset{\bullet}{x}, x \in X$ are compactoid, we have at first $\forall x \in X$: $\mu(\mathcal{F})(\overset{\bullet}{x}) \xrightarrow{q'_V(\sigma)} \mu(f)(\overset{\bullet}{x}) = f(\overset{\bullet}{x})$, thus from the definition of q'_V we get
$\forall \Psi \in \mathfrak{F}_0(\mu(\mathcal{F})(\overset{\bullet}{x})^\uparrow) : \forall A \in f(\overset{\bullet}{x}) : A \cap q_\sigma(\Psi^{\sqcup\!\sqcap}) \neq \emptyset$. Observe now, that $\mu(\mathcal{F})(\overset{\bullet}{x})$ is itself an ultrafilter on $\mathfrak{F}_0(Y)$ finer than $\mu(\mathcal{F})(\overset{\bullet}{x})^\uparrow$, because $\mathcal{F}$ is an ultrafilter and for each $F \in \mathcal{F}$, all singleton filters $g(\overset{\bullet}{x}), g \in F$, belong to $\mathfrak{F}_0(\bigcup_{g\in F} \mu(g)(\overset{\bullet}{x}))$. Taking $\{f(x)\}$ for A, we get then $\mu(\mathcal{F})(\overset{\bullet}{x})^{\sqcup\!\sqcap} \xrightarrow{\sigma} f(x)$ from the above. But it is easy to see, that $\mu(\mathcal{F})(\overset{\bullet}{x})^{\sqcup\!\sqcap} = \mathcal{F}(x)$, so $\mathcal{F}(x)$ converges to $f(x)$ for all $x \in X$ and consequently, $\mathcal{F}$ converges pointwise to f. Now, from the even continuity of $\mathcal{H}$ follows $(\mathcal{F}, f) \in q_c$.■

5.2 An Embedding Theorem for Multifilter - Spaces

In this section, we will try to apply our experiences from the foregoing, to derive an embedding- and then an Ascoli-like theorem for multifilter-spaces. The natural map between $\mathcal{H} \subseteq Y^X$ and $\mathfrak{P}_0(Y)^{\mathfrak{A}}$ for $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ is of pure set theoretical nature and therefore the same as before, for the beginning:

$$\mu : \mathcal{H} \to \mathfrak{P}_0(Y)^{\mathfrak{P}_0(X)} : f \to \mu(f) : A \to f(A) \ .$$

But now, we will restrict our observations only to $\mathfrak{A} := \mathcal{PC}(X)$ and $\mathcal{H}$ consisting of fine maps between multifilter-spaces $(X,\mathcal{M})$ and $(Y,\mathcal{N})$, thus μ maps such an $\mathcal{H}$ into $\mathcal{PC}(Y)^{\mathcal{PC}(X)}$, by corollary 69.

159 Proposition
Let $(X,\mathcal{M}),(Y,\mathcal{N})$ be limited multifilter-spaces and let f be a fine map from $(X,\mathcal{M})$ to $(Y,\mathcal{N})$. Then $\mu(f)$ is a fine map from $(\mathcal{PC}(X),\mathcal{M}_V)$ to $(\mathcal{PC}(Y),\mathcal{N}_V)$.

Proof: For $\Sigma\in\mathcal{M}$ we have $\mu(f)(\Sigma_V)=[\{\mu(f)(\alpha_V)|\ \alpha\in\Sigma\}]=$
$[\{\{\mu(f)(<A_1,...,A_n>)|\ n\in I\!N,A_1,...,A_n\in\alpha\}|\ \alpha\in\Sigma\}]$ and furthermore always $P\in<A_1,...,A_n>\Rightarrow P\in\bigcup_{i=1}^n A_i\wedge\forall i=1,...,n:P\cap A_i\neq\emptyset\Rightarrow f(P)\subseteq\bigcup_{i=1}^n f(A_i)\wedge\forall i=1,...,n:\emptyset\neq f(P\cap A_i)\subseteq f(P)\cap f(A_i)$, implying
$\mu(f)(<A_1,...,A_n>)\subseteq<f(A_1),...,f(A_n)>\in f(\alpha)_V\in f(\Sigma)_V$. Thus $\mu(f)(\Sigma_V)\preceq f(\Sigma)_V$, which belongs to $\mathcal{N}_V$, because f is fine. ■

160 Lemma
Let $(X,\mathcal{M}),(Y,\mathcal{N})$ be limited multifilter-spaces and $\mathcal{H}\subseteq Y^X$ a set of fine maps. Then μ is an injective and fine map from $(\mathcal{H},\mathcal{M}_{Y,pc})$ to $\mathcal{PC}(Y)^{\mathcal{PC}(X)}$, endowed with pointwise multifilter-structure, whenever $\mathcal{PC}(Y)$ is endowed with the hyperstructure $\mathcal{N}_V$.

Proof: Injectivity follows simply from the fact, that all singletons are always precompact. We have to show $\mu(\Gamma)(P)\in\mathcal{N}_V$ for every $P\in\mathcal{PC}(X)$ and $\Gamma\in\mathcal{M}_{Y,pc}$. So, let such Γ and P be given. $\Gamma\in\mathcal{M}_{Y,pc}$ just implies $\forall\Sigma_{|P},\Sigma\in\mathcal{M}:\Gamma(\Sigma)\in\mathcal{N}$, i.e.

$$\begin{aligned}\exists\Xi\in\mathcal{N}:\ \ \forall\xi\in\Xi:\exists\sigma\in\Sigma,\gamma\in\Gamma:\forall G\in\gamma,S\in\sigma:\\ \exists K_{S,G}\in\xi:\forall g\in G:g(S)\subseteq K_{S,G}\end{aligned}\tag{10}$$

Now, for precompact P from corollary 74 follows the existence of a $\Sigma_P\in\mathcal{M}$, s.t. $\forall\sigma\in\Sigma_P:\exists n_\sigma\in I\!N,S_1^{(\sigma)},...,S_{n_\sigma}^{(\sigma)}\in\sigma:\left(P\subseteq\bigcup_{i=1}^{n_\sigma}A_i^{(\sigma)}\wedge\forall i:P\cap A_i^{(\sigma)}\neq\emptyset\right)$ (the additional requirement of nonempty intersections is easy to ensure by just omitting all A's with empty intersection). Now, let Σ be the trace of Σ_P on P. Applying (10) to this Σ now yields

$$\begin{aligned}\exists\Xi\in\mathcal{N}:\forall\xi\in\Xi:\exists\sigma\in\Sigma,\gamma\in\Gamma:\ \ \forall G\in\gamma,S_i^{(\sigma)}\in\sigma:\exists K_{S_i^{(\sigma)},G}\in\xi:\\ \forall g\in G:g(S_i^{(\sigma)})\subseteq K_{S_i^{(\sigma)},G}\end{aligned}\tag{11}$$

Now, $g(S_i^{(\sigma)})\subseteq K_{S_i^{(\sigma)},G}$ implies $g(P)\subseteq g(\bigcup_{i=1}^{n_\sigma}S_i^{(\sigma)})=\bigcup_{i=1}^{n_\sigma}g(S_i^{(\sigma)})\subseteq\bigcup_{i=1}^{n_\sigma}K_{S_i^{(\sigma)},G}$ and of course $g(P)\cap K_{S_i^{(\sigma)},G}\supseteq g(P\cap S_i^{(\sigma)})\cap K_{S_i^{(\sigma)},G}=g(P\cap S_i^{(\sigma)})\neq\emptyset$ for each $i=1,...,n_\sigma$. Thus $\forall g\in G:g(P)\in<K_{S_1^{(\sigma)},G},...,K_{S_{n_\sigma}^{(\sigma)},G}>\in\xi_V$. Together with (11), this leads to $\mu(\gamma)(P)\preceq\xi_V$. Such a γ exists for all $\xi\in\Xi$, by (11), implying $\mu(\Gamma)(P)\preceq\Xi_V\in\mathcal{N}_V$, as desired. ■

Unfortunately, this doesn't work backwards without additional assumptions.

161 Corollary
Let $(X,\mathcal{M}),(Y,\mathcal{N})$ be limited multifilter-spaces and $\mathcal{H}\subseteq Y^X$ a set of fine maps. Then μ is an injective and fine map from $(\mathcal{H},\mathcal{M}_{\mathbf{X},\mathbf{Y}})$ to $\mathcal{PC}(Y)^{\mathcal{PC}(X)}$, endowed with pointwise multifilter-structure, whenever $\mathcal{PC}(Y)$ is endowed with the hyperstructure $\mathcal{N}_V$.

Proof: Follows from the lemma above and proposition 101. ■

Just the same procedure leads to a similar result, concerning precompact partial covers of X, instead of precompact subsets.

162 Lemma
Let $\mathcal{H}\subseteq Y^X$ consist of fine maps between the limited multifilter-spaces $(X,\mathcal{M})$ and $(Y,\mathcal{N})$. Then

$$\mu_2:(\mathcal{H},\mathcal{M}_{\mathbf{X},\mathbf{Y}})\to\mathcal{PC}(\mathcal{PC}(Y))^{\mathcal{PC}(\mathcal{PC}(X))}:f\to\mu_2(f):\alpha\to f(\alpha)$$

is fine and injective, where $\mathcal{PC}(\mathcal{PC}(Y))^{\mathcal{PC}(\mathcal{PC}(X))}$ is endowed with the pointwise multifilter-structure, generated from the hyperstructure $(\mathcal{N}_V)_V$ on $\mathcal{PC}(\mathcal{PC}(Y))$ and $\mathcal{PC}(X)$, $\mathcal{PC}(\mathcal{PC}(X))$ with the hyperstructures $\mathcal{M}_V$, $(\mathcal{M}_V)_V$, respectively.

Proof: That μ_2 is injective, follows simply from the fact, that the singletons $\{\{x\}\},x\in X$ are all contained in $\mathcal{PC}(\mathcal{PC}(X))$. We have to show, that $\mu_2(\Gamma)(\alpha)\in(\mathcal{N}_V)_V$ holds for all $\alpha\in\mathcal{PC}(\mathcal{PC}(X))$ and $\Gamma\in\mathcal{M}_{\mathbf{X},\mathbf{Y}}$. From $\Gamma\in\mathcal{M}_{\mathbf{X},\mathbf{Y}}$ we know again, that (10) holds, but now for all $\Sigma\in\mathcal{M}$. For precompact α we get from corollary 74, that there exists $\Sigma_\alpha\in\mathcal{M}$, s.t. $\forall\sigma\in\Sigma_\alpha:\exists n_\sigma,m_1,...,m_{n_\sigma}\in I\!N,S_1^{(1)},...,S_{m_{n_\sigma}}^{(n_\sigma)}\in\sigma:\alpha\subseteq\bigcup_{i=1}^{n_\sigma}<S_1^{(i)},...,S_{m_i}^{(i)}>$, i.e. $\forall P\in\alpha:\exists i_P\in\{1,...,n_\sigma\}:P\in<S_1^{(i_P)},...,S_{m_{i_P}}^{(i_P)}>$. (The $<S_1^{(i)},...,S_{m_i}^{(i)}>$ have to be chosen in a way, such that α meets each of them, which can be realized by simply omitting all others, again.) Applying (10) now, we get

$$\exists\Xi\in\mathcal{N}:\forall\xi\in\Xi:\exists\sigma\in\Sigma_\alpha,\gamma\in\Gamma:\ \ \forall G\in\gamma,S_i^{(j)}\in\sigma:\exists K_{S_i^{(j)},G}\in\xi:$$
$$\forall g\in G:g(S_i^{(j)})\subseteq K_{S_i^{(j)},G}\ , \qquad (12)$$

implying $\forall P\in\alpha,g\in G:g(P)\in<K_{S_1^{(i_P)},G},...,K_{S_{m_{i_P}}^{(i_P)},G}>$, thus $\forall g\in G:g(\alpha)\subseteq\bigcup_{i=1}^{n_\sigma}<K_{S_1^{(i)},G},...,K_{S_{m_i}^{(i)},G}>$ and $g(\alpha)$ meets all of the $<K_{S_1^{(i)},G},...,K_{S_{m_i}^{(i)},G}>$, because α meets all $<S_1^{(i)},...,S_{m_i}^{(i)}>$. So,

$$\forall g\in G:g(\alpha)\in\mathfrak{O}_G:=\left\langle<K_{S_1^{(1)},G},...,K_{S_{m_1}^{(1)},G}>,...,<K_{S_1^{(n_\sigma)},G},...,K_{S_{m_{n_\sigma}}^{(n_\sigma)},G}>\right\rangle$$

follows, implying $\mu_2(G)(\alpha)\subseteq\mathfrak{O}_G\in(\xi_V)_V$. But by (12), the existence of such an $\mathfrak{O}_G\in(\xi_V)_V$ follows for every $G\in\gamma$, implying $\mu_2(\gamma)\preceq(\xi_V)_V$, leading to $\mu_2(\Gamma)\preceq(\Xi_V)_V\in(\mathcal{N}_V)_V$, by regarding (12) again. ■

163 Lemma
Let $(X,\mathcal{M}),(Y,\mathcal{N})$ be limited multifilter-spaces, $\mathcal{H}$ a set of fine maps from X to Y, such that $\mu^{-1}:\mu(\mathcal{H})\to\mathcal{H}$ is fine, where $\mathcal{H}$ is endowed with the precompactly fine structure, and $\mu(\mathcal{H})$ with the pointwise structure w.r.t. $\mathcal{N}_V$ on $\mathcal{PC}(Y)$. Then are equivalent

(1) $\mathcal{H}$ is precompact w.r.t. the precompactly fine structure.

(2) For every $P\in\mathcal{PC}(X)$ is $\mathcal{H}(P):=\bigcup_{h\in\mathcal{H}}h(P)$ precompact in Y.

Proof: Let (1) be valid, then $\mu(\mathcal{H})$ is precompact by lemma 160 and corollary 69, and consequently for every $P\in\mathcal{PC}(X)$ is $\mu(\mathcal{H})(P)$ precompact w.r.t. $\mathcal{N}_V$, because it is the P-evaluation of $\mu(\mathcal{H})$. Now, the precompactness of $\mathcal{H}(P)$ follows from lemma 134.
Let otherwise (2) hold. Always $\mu(\mathcal{H})$ is naturally isomorphic to a subspace of $\prod_{P\in\mathcal{PC}(X)}(\mathfrak{P}_0(\mathcal{H}(P)),\mathcal{N}_{V|\mathfrak{P}_0(\mathcal{H}(P))})$, which is precompact by corollary 71, because all $\mathfrak{P}_0(\mathcal{H}(P))$ are precompact by theorem 133. Thus, $\mu(\mathcal{H})$ is precompact as isomorphic image of a subspace, and consequently $\mathcal{H}=\mu^{-1}(\mu(\mathcal{H}))$ is precompact by corollary 69, because μ^{-1} is fine by assumption. ■

164 Lemma
Let $(X,\mathcal{M}),(Y,\mathcal{N})$ be limited multifilter-spaces with $(Y,\mathcal{N})$ being weakly uniform and principal. Let $\mathcal{H}\subseteq Y^X$ be an equiuniformly fine family. Then $\mu^{-1}:\mu(\mathcal{H})\to\mathcal{H}$ is fine w.r.t. the pointwise structure on $\mu(\mathcal{H})$, generated from $\mathcal{N}_V$ on $\mathcal{PC}(Y)$, and the precompactly fine structure on $\mathcal{H}$.

Proof: Let $\mathcal{N}:=[\Xi]$ and let $\Sigma'\in\mathcal{M}$ with $P\in\mathcal{PC}(X)\cap(\Sigma')^{\cup}$ be given. Then there exists $\Sigma_1\in\mathcal{M}$, such that $\forall\sigma\in\Sigma_1:\exists n\in I\!N,S_1,...,S_n\in\sigma:P\subseteq\bigcup_{i=1}^n S_i$. Take $\Sigma:=\Sigma'\cap[\Sigma_{1\,|P}]\in\mathcal{M}$. Furthermore, let Γ be a multifilter on $\mathcal{H}$, s.t. $\mu(\Gamma)$ belongs to the pointwise multifilter-structure on $\mu(\mathcal{H})$.
Let $\xi\in\Xi$ be given.
At first, we know $[\mathcal{H}^{\{\}}](\Sigma)\preceq\Xi$, because $\mathcal{H}$ is equiuniformly fine. So, there exists $\sigma_0\in\Sigma$ with $\sigma_0^{\cup}=P$ and $\mathcal{H}^{\{\}}(\sigma_0)\preceq\xi$, and we know $\exists n\in I\!N,S_1,...,S_n\in\sigma_0:\bigcup_{i=1}^n S_i=P$. Each of these S_i is precompact, because P is, thus we have $\forall i=1,...,n:\mu(\Gamma)(S_i)\preceq(\Xi)_V$, because $\mu(\Gamma)\in\mathcal{M}_{\mathcal{N}_V,p}$. From this we get $\exists\gamma_i\in\Gamma:\mu(\gamma_i)(S_i)\preceq\xi_V$, for all $i=1,...,n$, i.e. $\forall G_i\in\gamma_i:\exists K^{(i)}_{1,G_i},...,K^{(i)}_{n_i,G_i}\in\xi:\mu(G_i)(S_i)\subseteq<K^{(i)}_1(G_i),...,K^{(i)}_{n_i}(G_i)>$, implying $G_i(S_i)\subseteq\bigcup_{j=1}^{n_i}K^{(i)}_{j,G_i}$. Take $\gamma:=\bigwedge_{i=1}^n\gamma_i$. Then we find $\forall G\in\gamma,i\in\{1,...,n\}:\exists G_i\in\gamma_i:G\subseteq G_i$, thus

$$\forall G\in\gamma,i\in\{1,...,n\}:G(S_i)\subseteq\bigcup_{j=1}^{n_i}K^{(i)}_{j,G_i}\ . \tag{13}$$

Now, we remember for arbitrary $G\in\gamma$ and $i\in\{1,...,n\}$ again $\exists G_i\in\gamma_i:G\subseteq G_i$, thus for an arbitrary element g_0 of any nonempty $G\in\gamma$ always holds $g\in G_i$,

just implying $\forall j = 1, ..., n_j : g(S_i) \cap K^{(i)}_{j,G_i} \neq \emptyset$ and from $\mathcal{H}^{\{\}}(\sigma_0) \preceq \xi$ we know $\exists K_0 \in \xi : g(S_i) \subseteq K_0$. But this yields $K_0 \cup \bigcup_{j=1}^{n_i} K^{(i)}_{j,G_i} \in \xi^{\diamond 2}$, thus

$$\gamma(\{S_1, ..., S_n\}) \preceq \xi^{\diamond 2} \ , \tag{14}$$

by (13). At least, let S be an arbitrary element of σ_0, $I_S := \{i \in I\!N |\ 1 \leq i \leq n, S_i \cap S \neq \emptyset\}$, then $S \subseteq \bigcup_{i \in I_S} S_i$. From (14) we know $\forall G \in \gamma, i \in I_S : \exists K_{G,i} \in \xi^{\diamond 2} : G(S_i) \subseteq K_{G,i}$, thus $G(S) \subseteq \bigcup_{i \in I_S} K_{G,i}$ and for an arbitrary element g_0 of the nonempty $G \in \gamma$ we have again $\forall i \in I_S : g_0(S) \cap g_0(K_{G,i}) \supseteq g_0(S) \cap g_0(S_i) \supseteq g_0(S \cap S_i) \neq \emptyset$, and from $\mathcal{H}^{\{\}}(\sigma_0) \preceq \xi \preceq \xi^{\diamond 2}$ we get again $K_S \in \xi^{\diamond 2}$ with $g_0(S) \subseteq K_S$, implying here $K_S \cup \bigcup_{i \in I_S} K_{G,i} \in (\xi^{\diamond 2})^{\diamond 2} = \xi^{\diamond 4}$. This is valid now for all $S \in \sigma_0$, thus $\gamma(\sigma_0) \preceq \xi^{\diamond 4}$, leading to $\Gamma(\Sigma) \preceq \Xi^{\diamond 4} = \Xi$, because from weak uniformity follows $\Xi^{\diamond 4} \in [\Xi]$, but $\Xi \preceq \Xi^{\diamond 4}$ by proposition 17(5). We started with Σ', but it's clear, that $\Sigma' \preceq \Sigma$, thus $\Gamma(\Sigma') \in \mathcal{N}$, too. ■

165 Corollary
Let $(X, \mathcal{M}), (Y, \mathcal{N})$ be limited multifilter-spaces with $(Y, \mathcal{N})$ being weakly uniform and principal. Let $\mathcal{H} \subseteq Y^X$ be a family of fine maps. Then the following are equivalent

(1) $\mathcal{H}$ is precompact w.r.t. the precompactly fine structure.

(2) (a) $\mathcal{H}$ is equiuniformly fine, and
(b) For every precompact subset $P \subseteq X$ is $\mathcal{H}(P) = \{h(p) |\ h \in \mathcal{H}, p \in P\}$ precompact in Y.

Proof: Follows immediately from the lemmata 164 and 163 and from lemma 97. ■

166 Corollary
Let $(X, \mathcal{M}), (Y, \mathcal{N})$ be limited multifilter-spaces with $(Y, \mathcal{N})$ being uniform and principal. Let $\mathcal{H} \subseteq Y^X$ be a family of fine maps. Then the following are equivalent

(1) $\mathcal{H}$ is precompact w.r.t. the precompactly fine structure.

(2) (a) $\mathcal{H}$ is equiuniformly fine, and
(b) For every $x \in X$ is $\mathcal{H}(x) = \{h(x) |\ h \in \mathcal{H}\}$ precompact in Y.

Proof: Combine proposition 99 with corollary 165. ■

Of course, if the domain space $(X, \mathcal{M})$ is assumed to be locally precompact, the foregoing statements concerning precompactness of $\mathcal{H}$ w.r.t. the precompactly fine structure hold w.r.t. the natural function space structure $\mathcal{M}_{\mathbf{X},\mathbf{Y}}$, because of proposition 101.

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Thesen zur Dissertation

von René Bartsch

1. Eine Teilmenge M eines topologischen Raumes (X, τ) heißt *schwach relativ vollständig* genau dann, wenn jeder Ultrafilter auf M, der in X konvergiert, auch in M konvergiert. Alle abgeschlossenen und alle kompakten Teilmengen sind schwach relativ vollständig. Ist (X, τ) ein topologischer Raum, besteht $\alpha \subseteq \mathfrak{P}_0(X)$ aus schwach relativ vollständigen Teilmengen von X und enthält $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ die (nichtleeren) abgeschlossenen Teilmengen, so ist $\mathfrak{A}$ bezüglich der mit α erzeugten hit-and-miss Topologie genau dann kompakt, wenn (X, τ) kompakt ist.

2. Ist eine Menge $\mathfrak{M}$ relativ kompakter Teilmengen eines topologischen Raumes (X, τ) relativ kompakt in der Menge aller relativ kompakten Teilmengen von X (bezüglich der oberen Vietoris-Topologie), so ist ihre Vereinigung relativ kompakt in X.

3. Seien $(X, \tau), (Y, \sigma)$ topologische Räume, $\mathcal{H} \subseteq Y^X$ und $\mathfrak{A}$ eine Teilmenge von $\mathfrak{P}_0(X)$, die die Einpunktmengen enthält. $\mathcal{H}$ sei mit der von $\mathfrak{A}$ erzeugten Mengen-offenen Topologie $\tau_{\mathfrak{A}}$ und $\mathfrak{P}_0(Y)^{\mathfrak{A}}$ mit der von der Vietoris-Topologie auf $\mathfrak{P}_0(Y)$ erzeugten punktweisen Topologie versehen. Dann ist die Abbildung $\mu : \mathcal{H} \to \mu(\mathcal{H}) := \{\mu(f)|\ \mu(f) : A \to f(A), f \in \mathcal{H}\} \subseteq \mathfrak{P}_0(Y)^{\mathfrak{A}}$ stetig, offen und bijektiv. Ist f eine stetige Funktion von X nach Y, so ist ihr Bild $\mu(f)$ stetig.

4. Seien $(X, \tau), (Y, \sigma)$ topologische Räume und $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$. Sei $\mathcal{H} \subseteq C(X, Y)$ und $\mathcal{F}$ ein Ultrafilter auf $\mathcal{H}$, der punktweise gegen eine Funktion $g \in C(X, Y)$ konvergiert. Dann gilt:

(a) Wenn $\mathfrak{A}$ aus relativ kompakten Teilmengen von X besteht, und $\mathcal{H}$ gleichstetig ist, sowie die Bilder aller Elemente von $\mathfrak{A}$ unter g abgeschlossen in Y sind, dann konvergiert $\mu(\mathcal{F})$ punktweise gegen $\mu(g)$.

(b) Wenn $\mathfrak{A}$ aus kompakten Teilmengen von X besteht und $\mathcal{H}$ gleichstetig auf allen Elementen von $\mathfrak{A}$ ist, so konvergiert $\mu(\mathcal{F})$ punktweise gegen $\mu(g)$.

5. Seien $(X, \tau), (Y, \sigma)$ topologische Räume und enthalte $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ die Einpunktmengen. Dann ist eine Teilmenge $\mathcal{H} \subseteq Y^X$ genau dann relativ kompakt in $(Y^X, \tau_{\mathfrak{A}})$, wenn

(a) für alle Ultrafilter $\mathcal{F}$ auf $\mathcal{H}$ mit $\mathcal{F} \xrightarrow{p} f \in Y^X$ eine Funktion $g \in Y^X$ existiert mit $\mu(\mathcal{F}) \xrightarrow{p} \mu(g) \in \mathfrak{P}_0(Y)^{\mathfrak{A}}$ und

(b) für alle $A \in \mathfrak{A}$ die Menge $\mu(\mathcal{H})(A) := \{f(A)|\ f \in \mathcal{H}\}$ relativ kompakt in $\mathfrak{P}_0(Y)$ bezüglich der Vietoris-Topologie ist.

6. **Ascoli-Satz:** Seien $(X,\tau),(Y,\sigma)$ topologische Räume, bestehe $\mathfrak{A}\subseteq\mathfrak{P}_0(X)$ aus relativ kompakten Teilmengen von X und enthalte die Einpunktmengen. Wenn $\mathcal{H}\subseteq C(X,Y)$ die Bedingungen

   (a) $\mathcal{H}$ ist schwach relativ vollständig in Y^X bezüglich punktweiser Konvergenz,

   (b) $\mathcal{H}$ ist gleichstetig,

   (c) die Bilder aller Elemente von $\mathfrak{A}$ unter Elementen von $\mathcal{H}$ sind abgeschlossen in Y und

   (d) für alle $x\in X$ ist $\mathcal{H}(x):=\{f(x_0)|\ f\in\mathcal{H}\}$ relativ kompakt in Y

   erfüllt, dann ist $\mathcal{H}$ kompakt bezüglich $\tau_{\mathfrak{A}}$. Sind alle Elemente von $\mathfrak{A}$ sogar kompakt, ist Bedingung (c) überflüssig und statt (b) genügt Gleichstetigkeit auf den Elementen von $\mathfrak{A}$.

7. Sei X eine Menge, $\mathcal{M}$ eine Menge von Filtern auf $\mathfrak{P}_0(X)$, dann heißt das geordnete Paar $(X,\mathcal{M})$ ein Powerfilter-Raum, falls alle von den Einpunktmengen $\{\{x\}\},x\in X$ erzeugten Filter zu $\mathcal{M}$ gehören und mit einem Filter $\Phi\in\mathcal{M}$ auch alle seine Oberfilter zu $\mathcal{M}$ gehören. Sind $(X,\mathcal{M})$ und $(Y,\mathcal{N})$ Powerfilter-Räume, so heißt eine Abbildung $f:X\to Y$ *fein*, falls $f(\mathcal{M})\subseteq\mathcal{N}$ gilt. Die Powerfilter-Räume und feinen Abbildungen bilden ein starkes topologisches Universum **PFS**.

8. Ein *Multifilter* auf einer Menge X is eine Familie Σ von Teilmengen von $\mathfrak{P}_0(X)$ (Teilüberdeckungen) mit den Eigenschaften, daß mit einem $\alpha\in\Sigma$ auch jede gröbere Teilüberdeckung zu Σ gehört und zu je zwei Elementen von Σ auch eine Teilüberdeckung zu Σ gehört, die feiner als beide ist. Ein *Multifilter-Raum* ist ein geordnetes Paar $(X,\mathcal{M})$ aus einer Menge X und einer Familie von Multifiltern auf X derart, daß alle von den Einpunktteilüberdeckungen $\{\{x\}\},x\in X$ erzeugten Multifilter zu $\mathcal{M}$ gehören und daß mit einem Multifilter $\Sigma\in\mathcal{M}$ auch jeder feinere Multifilter zu Σ gehört. Sind $(X,\mathcal{M})$ und $(Y,\mathcal{N})$ Multifilter-Räume, so heißt eine Abbildung $f:X\to Y$ *fein*, falls $f(\mathcal{M})\subseteq\mathcal{N}$ gilt. Die Multifilter-Räume und feinen Abbildungen bilden ein starkes topologisches Universum **MFS**, das konkret isomorph zur in **PFS** bireflektiven Unterkategorie $\mathbf{PFS}^{\preceq}$ der verfeinerungsabgeschlossenen Powerfilter-Räume ist. Limitierte, schwach uniforme, uniforme und Haupt-Multifilter-Räume bilden jeweils bireflektive Unterkategorien von **MFS**.

9. Die bireflektive Unterkategorie **PrULimMFS** (der uniformen Haupt-Multifilter-Räume) von **MFS** ist isomorph zur Kategorie der überdeckungsuniformen Räume im Sinne von Tukey. Auf Multifilter-Räumen $(X,\mathcal{M})$ sind eine Cauchy-Struktur $\gamma_{\mathcal{M}}$ und eine Konvergenz $q_{\gamma_{\mathcal{M}}}$ (damit auch Präkompaktheit, Kompaktheit und Vollständigkeit von Mengen) sowie für Funktionenmengen die

gleichgradige Feinheit erklärt, die im Falle der uniformen Haupt-Multifilter-Räume mit den entsprechenden Begriffen für die jeweils äquivalenten Tukey-Räume, übereinstimmen. $(X, q_{\gamma_{\mathcal{M}}})$ ist stets ein symmetrischer Kent-Konvergenzraum.

10. Hinsichtlich Präkompaktheit in Multifilter-Räumen gilt ein Tychonoff-Produktsatz.

11. Ein schwach uniformer limitierter Multifilter-Raum ist T_0 genau dann, wenn er T_2 ist und kompakt genau dann, wenn er präkompakt und vollständig ist.

12. Ist $(X, \mathcal{M})$ ein limitierter Multifilter-Raum, dann ist in Abhängigkeit von $\mathcal{M}$ eine (ebenfalls limitierte) Multifilter-Struktur $\mathcal{M}_V$ auf der Menge $\mathcal{PC}(X)$ der präkompakten Teilmengen von X erklärt. Eine Menge präkompakter Teilmengen von X ist präkompakt in Bezug auf $\mathcal{M}_V$ genau dann, wenn ihre Vereinigung präkompakt in Bezug auf $\mathcal{M}$ ist.

13. Neben der natürlichen Funktionenraumstruktur sind auf der Menge der feinen Abbildungen zwischen zwei Multifilter-Räumen weiterhin die punktweise und die präkompakt-feine Multifilter-Struktur erklärt; die präkompakt-feine stimmt bei lokal präkompaktem Urbildraum mit der natürlichen überein.

14. Seien $(X, \mathcal{M}), (Y, \mathcal{N})$ limitierte Multifilter-Räume und $\mathcal{H}$ eine Menge feiner Abbildungen von X nach Y. Dann besteht $\mu(\mathcal{H})$ für $\mu : \mathcal{H} \to \mathcal{PC}(Y)^{\mathcal{PC}(X)} : f \to \mu(f) : A \to f(A)$ aus feinen Abbildungen von $(\mathcal{PC}(X), \mathcal{M}_V)$ nach $(\mathcal{PC}(Y), \mathcal{N}_V)$ und μ ist injektiv und selbst fein hinsichtlich der präkompakt-feinen Struktur auf $\mathcal{H}$ und der punktweisen auf $(\mathcal{PC}(Y), \mathcal{N}_V)^{\mathcal{PC}(X)}$. Ist die inverse Abbildung μ^{-1} für gegebenes $\mathcal{H}$ ebenfalls fein, so sind äquivalent:

(a) $\mathcal{H}$ ist präkompakt bezüglich der präkompakt-feinen Struktur.

(b) Für alle präkompakten Teilmengen P von X ist $\mathcal{H}(P) := \{h(p) |\ h \in \mathcal{H}, p \in P\}$ präkompakt in Y.

15. Ist $(X, \mathcal{M})$ ein limitierter und $(Y, \mathcal{N})$ ein schwach uniformer Haupt-Multifilter-Raum, sowie eine Funktionenmenge $\mathcal{H} \subseteq Y^X$ gleichgradig fein, dann ist die inverse Abbildung $\mu^{-1} : \mu(\mathcal{H}) \to \mathcal{H}$ fein bezüglich der punktweisen Struktur auf $\mu(\mathcal{H})$ und der präkompakt-feinen auf $\mathcal{H}$.

16. Ist $(Y, \mathcal{N})$ in der Situation von 15 sogar uniform, dann ist

(c) Für alle $x \in X$ ist $\mathcal{H}(x) := \{h(x) |\ h \in \mathcal{H}\}$ präkompakt in Y.

äquivalent zu 14(b) und folglich wegen 14 und 15 zu 14(a), womit wir einen allgemeinen Ascoli-Satz haben.